Test 1 Solutions October 5, 2006

1. Determine whether the series converges or diverges, being sure to justify your claims. If possible, find the sum.

(a) $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \frac{80}{81} - \cdots$

This is a geometric series with $a = 5$ and $r = -2/3$. Because $|r| < 1$, it follows that the series converges to $\frac{5}{1+\frac{2}{3}} = 3.$

(b) $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}}$

Method I: We will attempt to compare this series to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series. Because our comparison series is divergent, we expect the given series to diverge, also. To prove this, we need the given series to be larger than the comparison series. Unfortunately, this is not the case ($\frac{1}{\sqrt{n+3}} < \frac{1}{\sqrt{n}}$). So we need to use the limit comparison test. For this, note that

$$\lim_{n\to\infty} \frac{1/\sqrt{n+3}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+3}} = \sqrt{\lim_{n\to\infty} \frac{1}{1+\frac{3}{n}}} = 1.$$

Because the limit is positive and finite, the two series are comparable. Since the comparison series diverges, this implies our series also diverges.

Method II: We use the integral test, noting that $f(x) = \frac{1}{\sqrt{x+3}}$ is a continuous, positive, decreasing function when $x \geq 0$. Thus we calculate

$$\int_0^{\infty} \frac{1}{\sqrt{x+3}}\, dx = \lim_{t\to\infty} \frac{1}{2}\sqrt{x+3}|_{x=0}^{t} = \frac{1}{2}(\lim_{t\to\infty} \sqrt{t+3} - \sqrt{3}) = \infty.$$

Because the integral diverges, the series also diverges.

(c) $\frac{1}{3} + \frac{\pi}{9} + \frac{\pi^2}{27} + \frac{\pi^3}{81} + \cdots$

This is a geometric series with $r = \pi/3 > 1$. It therefore diverges.

(d) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ (Hint: Write out the first few partial sums explicitly using the identity $\ln(a/b) = \ln(a) - \ln(b)$.)

Using the given identity on each summand, we calculate

$$s_1 = \ln(2) - \ln(1)$$
$$s_2 = (\ln(2) - \ln(1)) + (\ln(3) - \ln(2))$$
$$s_3 = (\ln(2) - \ln(1)) + (\ln(3) - \ln(2)) + (\ln(4) - \ln(3)),$$

so that, in general, we have

$$s_n = (\ln(2)-\ln(1))+(\ln(3)-\ln(2))+(\ln(4)-\ln(3))+\cdots+(\ln(n+1)-\ln(n)).$$

Canceling the telescoping terms, we find that

$$s_n = \ln(n+1) - \ln(1).$$

Thus

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (\ln(n+1) - \ln(1)) = \infty.$$

Thus the series diverges by definition, since the sequence of partial sums diverges.

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

We use the integral test. Note that $f(x) = \frac{1}{x\ln(x)}$ is a positive, continuous, decreasing function for $x \geq 2$. Thus we calculate

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty} \ln(\ln(t)) - \ln(\ln(2)) = \infty,$$

(use substitution with $u = \ln x$ and $du = dx/x$). It follows that the series diverges.

2. Consider the polar curve $r = 1 + 2\sin\theta$ (see the picture on the next page).

(a) Use the formulas relating polar to Cartesian coordinates to find a pair of parametric equations describing this curve (with parameter θ).

We use the equations $x = r\cos\theta$ and $y = r\sin\theta$, along with the fact that $r = 1 + 2\sin\theta$ to obtain

$$x = (1 + 2\sin\theta)\cos\theta \qquad y = (1 + 2\sin\theta)\sin\theta.$$

(b) On the picture, label the points corresponding to the values $\theta = 0$, $\theta = \pi/2$, $\theta = \pi$, and $\theta = 3\pi/2$.

The point where $\theta = 0$ is the rightmost point on the x-axis; the point where $\theta = \pi/2$ is the top of the larger loop; the point where $\theta = \pi$ is the leftmost point on the x-axis, and the point where $\theta = 3\pi/2$ is the top of the inner loop.

(c) Find the slope of the line tangent to this curve when $\theta = 0$.

We use the parametric equations found in part (a) to calculate

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos\theta\sin\theta + \cos\theta(1+2\sin\theta)}{2\cos^2\theta - \sin\theta(1+2\sin\theta)}.$$

When $\theta = 0$, this equals $\frac{1}{2}$.

(d) Set-up, but do not evaluate, an integral equal to the length of the inner loop of the curve. (Be careful about the limits of integration.)

For the limits of integration, we set $r = 0$ and solve for θ. In this case, we have $\sin\theta = -1/2$, from which we deduce that $\theta = 7\pi/6$ and $11\pi/6$. Using the parametric formula for arc-length (and the derivatives computed in part (c)), we obtain

$$\int_{7\pi/6}^{11\pi/6} \sqrt{[2\cos\theta\sin\theta + \cos\theta(1+2\sin\theta)]^2 + [2\cos^2\theta - \sin\theta(1+2\sin\theta)]^2}\, d\theta,$$

while the polar formula yields

$$\int_{7\pi/6}^{11\pi/6} \sqrt{(1+2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta.$$