

On all problems, “find the Taylor series” means either (i) find a summation notation formula, or (ii) find the first five non-zero terms explicitly.

1. For this problem you will use the fact that the Taylor series for e^x , centered at zero, is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

(a) Find the Taylor series for e^{-x} centered at zero.

We substitute $-x$ in for x in the original power series, to obtain

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots.$$

(b) The hyperbolic cosine function, or $\cosh(x)$, is defined to be

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}).$$

Use your answer to part (a) along with the given information to find the Taylor series for $\cosh(x)$ centered at zero.

Adding the series for e^x to that for e^{-x} , we see that the terms containing odd powers of x cancel, while the terms containing even powers of x double. Dividing by two, we are left with the even powers of x . Thus we have that

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots.$$

2. Find the interval of convergence for the following series (don't forget the endpoints):

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = 1 - \frac{3x}{\sqrt{2}} + \frac{9x^2}{\sqrt{3}} - \frac{27x^3}{\sqrt{4}} + \frac{81x^4}{\sqrt{5}} - \cdots.$$

Applying the ratio test, we have

$$\lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+1} x^{n+1} \sqrt{n+2}}{\sqrt{n+1} (-3)^n x^n} \right| = \lim_{n \rightarrow \infty} \left| 3x \sqrt{\frac{n+2}{n+1}} \right| = |3x|.$$

It follows that the series converges for x between $-1/3$ and $1/3$. We now check the endpoints. For $x = 1/3$, we obtain the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n (1/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}},$$

which converges by the alternating series test. For $x = -1/3$, we obtain

$$\sum_{n=0}^{\infty} \frac{(-3)^n (-1/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}},$$

which diverges by limit comparison with a p -series (for $p = 1/2$). Thus the final interval of convergence is

$$\left(-\frac{1}{3}, \frac{1}{3} \right].$$

3. Sketch the traces of the surface described by the equation below. Each trace graph should have at least three different curves on it.

$$x^2 + y - z^2 = 0$$

The x -traces are of the form $y = z^2 - k^2$, so these are parabolas opening upward, but shifting down as k moves away from zero. The y traces are of the form $x^2 - z^2 = -k$, so these are standard hyperbolas. The z -traces are of the form $y = -x^2 + k^2$, so these are parabolas opening upward, but shifting down as k moves away from zero.

Matching:

G: $x^2 + 4y^2 + 9z^2 = 1$

L: $y^2 = x^2 + z^2 - 1$

J: $y = 2x^2 + z^2$

A: $x^2 + 2z^2 = 1$

H: $y^2 = x^2 + z^2 + 1$

M: $y^2 = x^2 + 2z^2$

I: $y = x^2 - z^2$

C: $r = 2$

O: $\rho = 2$

B: $\theta = \pi/3$

F: $\phi = \pi/3$

4. Consider the vectors $\mathbf{a} = \langle 2, 1, 0 \rangle$ and $\mathbf{b} = \langle 1, -3, 1 \rangle$. Find the following:

(a) $\mathbf{a} \cdot \mathbf{b}$;

$$\mathbf{a} \cdot \mathbf{b} = (2)(1) + (1)(-3) + (0)(1) = -1$$

(b) $\mathbf{a} \times \mathbf{b}$;

$$\mathbf{a} \times \mathbf{b} = \langle 1, -2, -7 \rangle$$

(c) the area of the triangle with \mathbf{a} and \mathbf{b} as two of its sides

$$\text{Area} = \frac{1}{2} \|\mathbf{a} \times \mathbf{b}\| = \frac{3\sqrt{6}}{2}$$

(d) an equation for the plane through $(1, 2, 3)$ and parallel to both \mathbf{a} and \mathbf{b} ;

Parallel to both \mathbf{a} and \mathbf{b} means normal to $\mathbf{a} \times \mathbf{b}$. Using this normal vector found above and plugging the point in, we obtain

$$1(x - 1) - 2(y - 2) - 7(z - 3) = 0 \Rightarrow x - 2y - 7z = -24.$$

(e) parametric equations for the line of intersection between the plane through $(1, 2, 3)$ with normal vector \mathbf{a} and the plane through $(1, 2, 3)$ with normal vector \mathbf{b} ;

The fact that the line is contained in both planes means that it's perpendicular to both normals, and is thus parallel to $\mathbf{a} \times \mathbf{b}$, found above. Since both planes go through $(1, 2, 3)$, the line must also go through $(1, 2, 3)$. Plugging in, we obtain

$$\mathbf{r}(t) = \langle 1 + t, 2 - 2t, 3 - 7t \rangle.$$