

Mini-Test 2 Solutions

Calculus II

October 26, 2007

1. Evaluate the following integral: $\int_0^1 x \ln(x) dx$

The integrand is not defined at $x = 0$, so this is an improper integral. We thus have

$$\int_0^1 x \ln(x) dx = \lim_{t \rightarrow 0^+} \left[\int_t^1 x \ln(x) dx \right].$$

We evaluate this integral using integration by parts, setting $u = \ln(x)$ and $dv = x$. This gives

$$\begin{aligned} \int x \ln(x) dx &= \int u dv = uv - \int v du = \frac{x^2}{2} \ln(x) - \int \frac{x^2}{2x} dx \\ &= \frac{x^2}{2} \ln(x) - \frac{1}{2} \int x dx = \frac{x^2}{2} \ln(x) - \frac{x^2}{4}. \end{aligned}$$

Evaluating from t to 1, we have

$$\left(\frac{1}{2} \ln(1) - \frac{1}{4} \right) - \left(\frac{t^2}{2} \ln(t) - \frac{t^2}{4} \right) = -\frac{1}{4} - \frac{1}{2} t^2 \ln(t) + \frac{t^2}{4}.$$

We now take the limit as $t \rightarrow 0^+$. Clearly, we have $\lim_{t \rightarrow 0} \frac{t^2}{4} = 0$, but for the other term, we need to use l'Hospital's rule.

$$\lim_{t \rightarrow 0^+} t^2 \ln(t) = \lim_{t \rightarrow 0^+} (2t) \left(\frac{1}{t} \right) = \lim_{t \rightarrow 0^+} 2 = 2.$$

So we finally have

$$\int_0^1 x \ln(x) dx = -\frac{1}{4} - \frac{1}{2}(2) + 0 = -\frac{5}{4}.$$

2. Write out the form of the partial fraction decomposition of the function below. Do not integrate anything, and do not solve for anything.

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2 + x + 1)(x^2 + 1)^3}$$

$$\frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

3. Consider the four-leaved rose given in polar coordinates by $r = \cos(2\theta)$.

(a) Find the Cartesian coordinates of the point where $\theta = \pi/3$.

If $\theta = \pi/3$, then $r = \cos(2\pi/3) = -1/2$. Then

$$x = r \cos(\theta) = -\frac{1}{2} \cos(\pi/3) = -\frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{4}$$

and

$$y = r \sin(\theta) = -\frac{1}{2} \sin(\pi/3) = -\frac{1}{2} \cdot \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{4}.$$

(b) Find the slope of the tangent line at that point.

Using the parametric expressions (from the next part), we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2 \sin(2t) \sin(t) + \cos(2t) \cos(t)}{-2 \sin(2t) \cos(t) - \cos(2t) \sin(t)}.$$

Evaluating at $t = \pi/3$, we have

$$\frac{dy}{dx} = \frac{-2 \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2} - \frac{1}{2} \frac{1}{2}}{-2 \frac{\sqrt{3}}{2} \frac{1}{2} + \frac{1}{2} \frac{\sqrt{3}}{2}} = \frac{-\frac{3}{2} - \frac{1}{4}}{-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}} = \frac{-7/4}{-\sqrt{3}/4} = \frac{7\sqrt{3}}{3}.$$

(c) Find a Cartesian parametrization of the curve.

$$x = r \cos(\theta) = \cos(2\theta) \cos(\theta) = \cos(2t) \cos(t)$$

$$y = r \sin(\theta) = \cos(2\theta) \sin(\theta) = \cos(2t) \sin(t)$$

(d) Find an integral expression equal to the length of one petal of the rose. Do not evaluate the integral.

Using the polar form of the arc-length integral, we have

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_{-\pi/4}^{\pi/4} \sqrt{\cos^2(2\theta) + 4 \sin^2(2\theta)} d\theta.$$

If we use the parametric form instead, we obtain

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt$$
$$= \int_{-\pi/4}^{\pi/4} \sqrt{\left(-2 \sin(2t) \sin(t) + \cos(2t) \cos(t)\right)^2 + \left(-2 \sin(2t) \cos(t) - \cos(2t) \sin(t)\right)^2} dt$$

which eventually simplifies to the polar expression above.