Test 1 Solution

Calculus II **September 28, 2007**

1. Consider a pool of the shape indicated below filled with water. Do one, and only one, of the following:
(a) Find the work done in pumping the water out of the pool through the spout.
(b) Find the force exerted by the water on one of the triangular walls.

For both problems, we need to find the area of a generic slice. Putting the origin at the top of the tank and pointing the axis downward, a horizontal slice at depth x along a triangular side of the pool has length $\frac{2}{3}(3-x) = 2 - \frac{2}{3}x$.

For work, we multiply this quantity by 8 to get the area of a slice, multiply by Δx to get the volume of a thickened slice, multiply by 1000 to get the mass of such a slice, multiply by 9.8 to get the force required to lift such a slice, and multiply by the distance (which here is depth, x) to get the work done in lifting such a slice to the top of the pool. Thus the work for a slice is $(9.8)(1000)(8)x(2-\frac{2}{3}x)\Delta x$, and the total work is

$$W = (9.8)(1000)(8)\int_0^3 (2x - \frac{2}{3}x^2)\,dx = (9.8)(1000)(8)\left[x^2 - \frac{2x^3}{9}\right]_{x=0}^{x=3}$$

$$= (9.8)(1000)(8)[9-6] = (9.8)(1000)(8)(3) = 235,200.$$

For hydrostatic force, we multiply by Δx to get the area of a small horizontal strip on the triangle. For the pressure along this strip, we multiply the density of water (1000), gravity (9.8), and depth (x). Thus the force on such a slice is $(9.8)(1000)x(2-\frac{2}{3}x)\Delta x$. Thus the total force is

$$F = (9.8)(1000)\int_0^3 (2x - \frac{2}{3}x^2)\,dx = (9.8)(1000)(3) = 29,400,$$

where we saved some work by noticing that this is the same integral as that for work, except without the factor of 8.

2. Evaluate the following integrals:

$$\int_0^2 \frac{2x}{(x-3)^2}\,dx$$

We use substitution, with $u = x - 3$, so that $du = dx$, and $x = u + 3$. Making these substitutions, we have

$$\int_0^2 \frac{2x}{(x-3)^2}\,dx = \int_{-3}^{-1} \frac{2(u+3)}{u^2}\,du = \int_{-3}^{-1} \left(\frac{2}{u} + \frac{6}{u^2}\right) du$$

$$= 2\ln|u| - \frac{6}{u} = 6 - (2\ln(3) - 2) = 8 - 2\ln(3).$$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx$$

We use the trig substitution $x = 3\sin(u)$, so that $dx = 3\cos(u)\,du$. Plugging this substitution in, we obtain

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2(u)}{3\cos(u)} 3\cos(u)\,du = 9\int \sin^2(u)\,du$$

$$= 9\left(\frac{1}{2}u - \frac{1}{2}\sin(u)\cos(u)\right) + C = \frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\sin^{-1}(x) - \frac{1}{2}x\sqrt{9-x^2} + C.$$

3. Let $\mathcal{R}$ denote the region bounded by the graph of $y = x^2$ and the line $y = 4$. Express each of the following quantities as an integral in a single variable. Do not evaluate the integrals.

(a) The area of $\mathcal{R}$.

$$A = \int_{-2}^{2} (4 - x^2)\,dx$$

(b) The volume of the solid obtained by rotating $\mathcal{R}$ about the x-axis.

$$V = \int_{-2}^{2} \pi(16 - x^4)\,dx$$

(c) The volume of the solid obtained by rotating $\mathcal{R}$ about the line $x = 3$.

$$V = \int_{-2}^{2} 2\pi(3 - x)(4 - x^2)\,dx$$