

Test 2 Solutions

Calculus II

November 9, 2007

1. Evaluate the following integral: $\int_0^{\infty} xe^{-x} dx$

We first write

$$\int_0^{\infty} xe^{-x} dx = \lim_{t \rightarrow \infty} \int_0^t xe^{-x} dx.$$

To evaluate this integral, we use parts, with $u = x$ and $dv = e^{-x} dx$. Thus we have

$$\int xe^{-x} dx = -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x}.$$

Evaluating from zero to t we have

$$\int_0^t xe^{-x} dx = [-xe^{-x} - e^{-x}]_{x=0}^{x=t} = (-te^{-t} - e^{-t}) - (0 - 1) = 1 - e^{-t} - te^{-t}.$$

Taking the limit as $t \rightarrow \infty$, this becomes

$$\int_0^{\infty} xe^{-x} dx = \lim_{t \rightarrow \infty} (1 - e^{-t} - te^{-t}) = 1 - 0 - 0 = 1,$$

where the limit of te^{-t} (an indeterminate form of type $0 \cdot \infty$) is obtained using l'Hospital's rule:

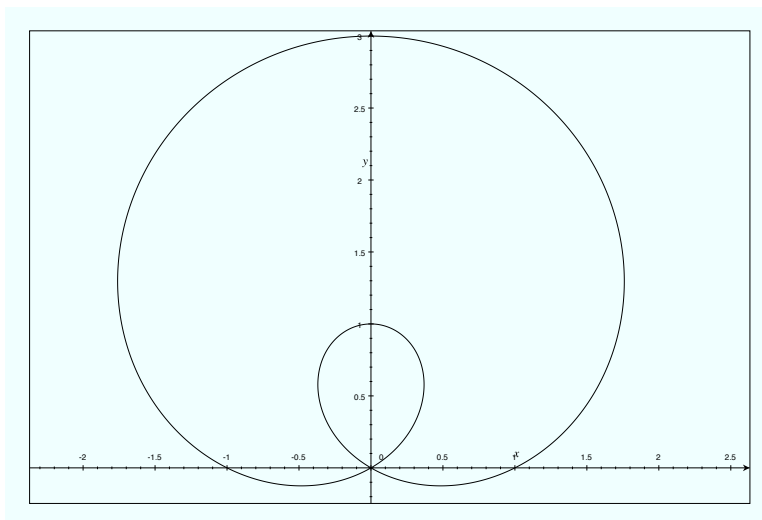
$$\lim_{t \rightarrow \infty} (t)(e^{-t}) = \lim_{t \rightarrow \infty} (1)(-e^{-t}) = 0.$$

2. Write out the form of the partial fraction decomposition of the function below. Do not integrate anything, and do not solve for anything.

$$\frac{x^3 + x^2 + 1}{x^2(x-1)^2(x^2+1)}$$

$$\frac{x^3 + x^2 + 1}{x^2(x-1)^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{Ex+F}{x^2+1}$$

3. Consider the circle given in polar coordinates by $r = 1 + 2 \sin(\theta)$ shown below.



- (a) On the picture, label the points corresponding to the values $\theta = 0$, $\theta = \pi/2$, $\theta = \pi$, and $\theta = 3\pi/2$.

The point where $\theta = 0$ is where the curve crosses the positive x -axis, at Cartesian point $(1,0)$. The point where $\theta = \pi/2$ is the top of the curve, at the Cartesian point $(0,3)$. The point at which $\theta = \pi$ is where the curve crosses the negative x -axis, at the Cartesian point $(-1,0)$. The point at which $\theta = 3\pi/2$ is the top of the smaller loop, at the Cartesian point $(0,1)$.

- (b) Find a Cartesian parametrization of the curve.

We use the equations $x = r \cos \theta$ and $y = r \sin \theta$ to obtain

$$x = (1 + 2 \sin(\theta)) \cos(\theta) \quad y = (1 + 2 \sin(\theta)) \sin(\theta).$$

- (c) Find the slope of the tangent line at the point where $\theta = 0$.

Using the parametrization above, we have

$$\frac{dy}{d\theta} = 2 \cos \theta \sin \theta + (1 + \sin \theta) \cos \theta = (3 \sin \theta + 1) \cos \theta$$

and

$$\frac{dx}{d\theta} = 2 \cos \theta \cos \theta - (1 + 2 \sin \theta) \sin \theta = 2(\cos^2 \theta - \sin^2 \theta) - \sin \theta.$$

Plugging in $\theta = 0$ we have

$$\frac{dy}{dx} = \frac{(3 \sin(0) + 1) \cos(0)}{2(\cos^2(0) - \sin^2(0)) - \sin(0)} = \frac{1}{2}.$$

- (d) Find an integral expression equal to the length of the inner loop of the curve. Do not evaluate the integral.

First we solve for the limits of integration. We want to start and stop measuring at the origin, so we solve for when $r = 0$. This happens when $1 + 2\sin\theta = 0$, or $\sin\theta = -\frac{1}{2}$. Looking at the pictures, we see that integral should go from $7\pi/6$ to $11\pi/6$. Then using the arclength formula for polar coordinates we have

$$\begin{aligned} L &= \int_{7\pi/6}^{11\pi/6} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_{7\pi/6}^{11\pi/6} \sqrt{(1 + 2\sin\theta)^2 + (2\cos\theta)^2} d\theta \\ &= \int_{7\pi/6}^{11\pi/6} \sqrt{5 + 4\sin\theta} d\theta. \end{aligned}$$

- (e) Find an integral expression equal to the area inside the inner loop of the curve. Do not evaluate the integral.

We use the same limits of integration as above, but integration using the polar area formula. Thus we have

$$A = \int_{7\pi/6}^{11\pi/6} \frac{1}{2}(1 + 2\sin\theta)^2 d\theta.$$

4. (a) Write the formula for the Taylor series for a function $f(x)$ centered at $x = 0$.

The formula is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

- (b) Use a known Taylor series to find the Taylor series for $f(x) = \int \frac{1}{1 - 2x^3} dx$ centered at zero.

We start with the known series

$$\frac{1}{1 - x} \leftrightarrow \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots,$$

and substitute $2x^3$ in for x , obtaining

$$\frac{1}{1 - 2x^3} \leftrightarrow \sum_{n=0}^{\infty} (2x^3)^n = \sum_{n=0}^{\infty} 2^n x^{3n} = 1 + 2x^3 + 4x^6 + 8x^9 + 16x^{12} + \dots$$

Now we integrate term-by-term, obtaining

$$\begin{aligned}\int \frac{1}{1-2x^3} dx &\leftrightarrow \sum_{n=0}^{\infty} \int 2^n x^{3n} dx = \sum_{n=0}^{\infty} \frac{2^n}{3n+1} x^{3n+1} \\ &= x + \frac{2}{4}x^4 + \frac{4}{7}x^7 + \frac{8}{10}x^{10} + \frac{16}{13}x^{13} + \dots\end{aligned}$$

(c) Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n3^n}$

We use the ratio test, obtaining

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1} n 3^n}{(n+1) 3^{n+1} x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{3} \frac{n}{n+1} \right| = \frac{|x|}{3}.$$

Setting this less than one, we find that we have

$$\frac{|x|}{3} < 1 \quad \Rightarrow \quad |x| < 3 \quad \Rightarrow \quad -3 < x < 3.$$

Thus the radius of convergence is half the length of this interval, which is three.