Final Study Guide

All sets and sequences consist of real numbers, and all functions have subsets of the real numbers as domain and range. The term "closed interval" means an interval of the form $[a, b] \subset \mathbf{R}$, where a and b are finite real numbers. Riemann integration is always assumed to take place over finite intervals $[a, b]$. Statements of theorem below are sloppy and inaccurate; you should know the precise statements referenced in the text.

DEFINITIONS

- Two sets have the same cardinality if
- A set is countable if
- A set is uncountable if
- A sequence converges if
- A sequence is bounded if
- A sequence is monotone if
- A sequence is Cauchy if
- A set is complete if
- The supremum of a set is
- The infimum of a set is
- A subsequence of a sequence is
- A number is an accumulation point of a sequence if
- A function is continuous at a point if (epsilon-delta version)
- A function is bounded if
- A function is uniformly continuous if
- A function is Lipschitz continuous if
- A function is Riemann integrable if
- A function is differentiable if
- The nth Taylor polynomial of an n-times differentiable function is

THEOREMS

- (1.3.5) The rationals are countable
- (1.3.6) The reals are uncountable
- (2.2.2) The squeeze theorem
- (2.2.3-6) Algebra of limits of sequences
- (2.4.1+Axiom of Completeness) A sequence converges if and only if it is Cauchy
- (2.4.3) Bounded monotone sequences converge
- (2.5.1) Sups and infs exist and are unique
- (2.6.1) Accumulation points are exactly limits of subsequences
- (2.6.2: Bolzano-Weierstrass) Every bounded sequence has at least one convergent subsequence
- (3.1.1-2) Algebra of continuous functions
- (3.1.3) Equivalence of the two definitions of continuity (epsilon-delta versus limit)
- (3.2.1) A continuous function on a closed interval is bounded
- (3.2.2) A continuous function on a closed interval achieves its inf and sup
- (3.2.3: IVT) A continuous function on a closed interval achieves every value between its inf and sup
- (3.2.5) A continuous function on a closed interval is uniformly continuous
- (3.2#7) Lipschitz continuous implies uniformly continuous
- (Lemma 3) If partitions exist that make the upper and lower sums of a function arbitrarily close, then the function is Riemann integrable
- (3.3.1) Continuous functions are Riemann integrable
- (3.3.3) Algebra of Riemann integrals
- (3.5.1) Bounded monotone functions are Riemann integrable
- (3.5.3) Bounded functions with finitely many discontinuities are Riemann integrable
- (4.1.1) Differentiable at a point implies continuous at that point
- (4.1.2) Algebra of derivatives
- (4.1.3) The chain rule

- (4.2.1) If a function is differentiable at its maximum, then its derivative vanishes there
- (4.2.2: Rolle's) If a differentiable function on an interval is zero at its endpoints, then its derivative is zero at some interior point
- (4.2.3: MVT) For a differentiable function on an interval, there is some interior point where the instantaneous rate of change equals the average rate of change over the interval
- (4.2.4: FTC I) $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$
- (4.2.5: FTC II) $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$
- (4.3.1: Taylor's) $f(x) = T_n(x, x_0) + \dfrac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$
- (4.3.2: l'Hospital's) $\displaystyle\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$

EXAMPLES

- An infinite countable set
- An uncountable set
- A bounded sequence that does not converge
- A monotone sequence that does not converge
- A sequence that does not contain its sup
- A sequence with multiple accumulation points
- A sequence with infinitely many accumulation points
- An unbounded function on a closed interval
- A function on a closed interval that does not achieve its sup
- A function not satisfying the intermediate value theorem
- A continuous but non-uniformly continuous function
- A uniformly continuous but not Lipschitz continuous function
- A non-Riemann integrable function
- A continuous but non-differentiable function