Test 1 Solutions

Advanced Calculus I **September 26, 2007**

1. Use the definition of convergence to show that the sequence $a_n = \sqrt{1 + \frac{1}{n}}$ converges.

Note that the given sequence converges to 1. Let $\epsilon > 0$ be given. We need to find N large enough so that

$$\left| \sqrt{1 + \frac{1}{n}} - 1 \right| < \epsilon.$$

First, note that we may drop the absolute value bars, as the expression inside is always positive. Solving the resulting expression for n, we find

$$\begin{aligned} & \sqrt{1 + \frac{1}{n}} - 1 < \epsilon \\ \Leftrightarrow \quad & \sqrt{1 + \frac{1}{n}} < \epsilon + 1 \\ \Leftrightarrow \quad & 1 + \frac{1}{n} < \epsilon^2 + 2\epsilon + 1 \\ \Leftrightarrow \quad & \frac{1}{n} < \epsilon^2 + 2\epsilon \\ \Leftrightarrow \quad & n > \frac{1}{\epsilon^2 + 2\epsilon}. \end{aligned}$$

Thus if we choose $N > \frac{1}{\epsilon^2 + 2\epsilon}$, then for $n \geq N$, we have

$$\left| \sqrt{1 + \frac{1}{n}} - 1 \right| \leq \left| \sqrt{1 + \frac{1}{N}} - 1 \right| < \left| \sqrt{1 + (\epsilon^2 + 2\epsilon)} - 1 \right| = \epsilon.$$

2. Do one of the following definition/theorem pairs, but not both. In either case, complete the definition, then prove the theorem.

Defn: A sequence $\{a_n\}$ *converges* to a limit a if, for all $\epsilon > 0$, ...
Thm: If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty} (a_n + b_n) = a + b$.

A sequence $\{a_n\}$ converges to a limit a if, for all $\epsilon > 0$, there is some N (depending on ϵ) so that $|a_n - a| < \epsilon$ for all $n \geq N$.

Let $\epsilon > 0$ be given. We need to find some N so that, for all $n \geq N$, we have $|(a_n + b_n) - (a + b)| < \epsilon$.

Because $a_n \to a$, there is some N_1 so that $|a_n - a| < \frac{\epsilon}{2}$ whenever $n \geq N_1$. Similarly, there is some N_2 so that $|b_n - b| < \frac{\epsilon}{2}$ whenever $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$, we have

$$|(a_n+b_n)-(a+b)| = |(a_n-a)+(b_n-b)| \leq |a_n-a|+|b_n-b| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon,$$

where the first inequality is the triangle inequality.

Defn: A sequence $\{a_n\}$ is *Cauchy* if, for all $\epsilon > 0$, ...
Thm: If a sequence converges, then the sequence is Cauchy.

A sequence $\{a_n\}$ is Cauchy if, for all $\epsilon > 0$< there is some N (depending on ϵ) so that $|a_n - a_m| < \epsilon$ whenever both $n > N$ and $m > N$.

Let $\epsilon > 0$ be given. We need to find some N so that, whenever $n \geq N$ and $m \geq N$, we have $|a_n - a_m| < \epsilon$.

Suppose $a_n \to a$. Then there is some N so that $|a_n - a| < \frac{\epsilon}{2}$ whenever $n \geq N$. Thus for $n \geq N$ and $m \geq N$ we have

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

where the first inequality is the triangle inequality.

3. Suppose you are given a sequence $\{a_n\}$ so that, for some $r > 0$, we have $0 < a_1 < \frac{1}{r}$ and $a_{n+1} = a_n(2 - ra_n)$ for $n \geq 1$.

- Consider the function $y = x(2 - rx)$. Show that for this function, $0 < x < \frac{1}{r}$ implies $0 < y < \frac{1}{r}$. Deduce that $0 < a_n < \frac{1}{r}$ for all n. Hint: What does the graph of this function look like? Where does it have a maximum? What is that maximum?

  This function is $y = 2x - rx^2$. The graph is a parabola opening down. This function has two roots, namely $x = 0$ and $x = 2/r$. It follows that the unique maximum occurs when $x = 1/r$, at which point we have $y = 1/r$. (One can deduce this using calculus techniques, also.) It follows that whenever $0 < x < 1/r$, we also have $0 < y < 1/r$. From this we deduce that $0 < a_n < 1/r$ implies that $0 < a_{n+1} < 1/r$, as required. As $0 < a_1 < 1/r$, it follows inductively that $0 < a_n < 1/r$ for all n.

- Use the bounds above to show that this sequence is monotone increasing. In other words, show that $a_{n+1} \geq a_n$.

For this we have

$$a_{n+1} = a_n(2 - ra_n) > a_n\left(2 - r\frac{1}{r}\right) = a_n,$$

so that the sequence is monotone increasing.

- State the theorem that allows you to deduce that the sequence converges.

  The bounded monotone convergence theorem states that any sequence that is both monotone and bounded must converge.

4. A real number is called *algebraic* if it is the root of some polynomial with integer coefficients. Otherwise, it is called *transcendental*. In this problem, we will show that the set of algebraic numbers is countable.

- Complete the following definition: A set is *countable* if ...

  A set S is countable if there is some one-to-one function $f : S \to \mathbf{N}$.

- Let P_n denote the set of all polynomials of degree n with integer coefficients. Thus P_n is the set of all expressions of the form $a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$, where the a_i are all integers. Show that P_n is countable by defining a 1–1 function from P_n to the $(n+1)$-fold product $\mathbf{Z} \times \mathbf{Z} \times \cdots \times \mathbf{Z}$.

  Set $f : P_n \to \mathbf{Z}^{n+1}$ to be

$$f(a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0) = (a_n, \ldots, a_2, a_1, a_0).$$

  That this function is one-to-one follows from the fact that two polynomials are equal precisely if their corresponding coefficients are equal. This shows that the cardinality of P_n is no more than that of $\mathbf{Z}^{n+1}$, which is countable, being a finite Cartesian product of countable sets.

- Use the previous fact to argue that the set P of *all* polynomials with integer coefficients (of any degree) is countable. Feel free to cite a theorem.

  Note that P is the union, over all $n \in \mathbf{N}$, of the sets P_n. Thus P is a countable union of countable sets, and hence is countable.

- Finally, use the fact that a polynomial of degree n has at most n distinct roots to deduce that the set of all algebraic numbers is countable.

  Because the set of all polynomials with integer coefficients is countable, and each polynomial has only finitely many roots, the set of all roots of such polynomials is the countable union of finite sets, hence countable.