Test 2 Solutions

Advanced Calculus I **November 7, 2007**

1. (a) State the Bolzano-Weierstrass theorem.

`Every bounded sequence has a convergent subsequence.`

(b) Give an example of a bounded sequence that does not converge to a limit.

$a_n = (-1)^n$

(c) Carefully define the *supremum* of a set.

The ***supremum*** of a set S is any number b so that (i) $s \leq b$ for all $s \in S$ (i.e., b is an upper bound for S), and (ii) if b' is any other upper bound, then $b \leq b'$ (i.e., b is a least upper bound).

(d) Suppose $\{a_1, a_2, a_3, \ldots\}$ is a bounded sequence of real numbers, and suppose we define $s_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$. Show that this sequence is monotonically decreasing and bounded.

To see that $\{s_n\}$ is monotonically decreasing, note that if $S' \subseteq S$, then $\sup S' \leq \sup S$. ($\sup S'$ is the least upper bound of a smaller set of numbers; by getting rid of numbers to consider, the sup can only get lower; if this is not clear, write down some explicit examples). The fact that $\{s_n\}$ is bounded follows immediately from the fact that the a_n are bounded (any lower bound for the a_n is also a lower bound for the s_n).

(e) State carefully the theorem that allows you to deduce that this "sequence of sups" $\{s_n\}$ converges.

`A bounded, monotone sequence converges.`

(f) Show that $\lim_{n\to\infty} s_n = S$ is a limit point (accumulation point) of $\{a_i\}$.

Let $\epsilon > 0$ be given. We will show that the ϵ-neighborhood of S contains infinitely many of the a_i. Because $s_n \to S$, there is some N so that $|S - s_n| < \epsilon/2$ whenever $n \geq N$. Because $s_n = \sup\{a_n, a_{n+1}, \ldots\}$, there is some M so that $|s_n - a_m| < \epsilon/2$ whenever $m \geq M$ (otherwise s_n wouldn't be a *least* upper bound). By the triangle inequality, it follows that $|a_m - S| < \epsilon$ whenever $m \geq \max\{M, N\}$.

(g) Show that S is the largest limit point of $\{a_i\}$.

Suppose there is some limit point A of $\{a_i\}$ with $A > S$. Let $\epsilon = \frac{1}{3}(A - S)$. Because S is the limit of the s_n, there is some s_j with $|s_j - S| < \epsilon$. In particular, we have $s_j < S + \epsilon$. But s_j is the supremum of the set $\{a_j, a_{j+1}, \ldots\}$. This means that, for all but finitely many i, we have $a_i < s_j < S + \epsilon$. On the other hand, because A is a limit point, there are infinitely many i so that $|a_i - A| < \epsilon$. In particular, there are infinitely many i so that $a_i > A - \epsilon > S + \epsilon$. This is a contradiction.

2. (a) Define what it means for a function to be uniformly continuous.

A function f is uniformly continuous on some domain D if, for all $\epsilon > 0$, there is some $\delta > 0$ so that for all x and y in D, we have $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$.

(b) Show that if f and g are uniformly continuous, then so is $f \circ g$.

Let $\epsilon > 0$ be given. Because f is uniformly continuous, there is some $\eta > 0$ so that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \eta$. But because g is uniformly continuous, there is some $\delta > 0$ so that $|g(x) - g(y)| < \eta$ whenever $|x - y| < \delta$. Thus for $|x - y| < \delta$, we have $|f(g(x)) - f(g(y))| < \epsilon$.

(c) Define $f : (0, 1] \to \mathbf{R}$ by $f(x) = x \sin \frac{1}{x}$. Show that if we define $f(0) = 0$, the resulting function $f : [0, 1] \to \mathbf{R}$ is continuous at $x = 0$.

Let $\{x_n\}$ be any sequence of real numbers with $x_n \to 0$. Note that $0 < |x_n \sin \frac{1}{x_n}| \le |x_n|$. It follows from the squeeze theorem that $|x_n \sin \frac{1}{x_n}| \to 0$, and so $x_n \sin \frac{1}{x_n} \to 0$. Thus f is continuous at zero if we define $f(0) = 0$.

(d) Carefully state a theorem that allows you to deduce that $f : [0, 1] \to \mathbf{R}$ is uniformly continuous.

A function that is continuous on a closed and bounded interval is uniformly continuous on that interval.

(e) Define what it means for a function to be Lipschitz continuous.

A function f is Lipschitz continuous if there is some M so that $\dfrac{|f(x) - f(y)|}{|x - y|} \le M$ for all x and y in the domain.

(f) Show that if g is a differentiable function that is Lipschitz continuous, then the derivative of g is bounded; i.e., there is some constant M so that $|g'(x)| \leq M$ for all x in the domain of g. Recall that the derivative of g is defined by

$$g'(x) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}.$$

Let $M \geq 0$ be the Lipschitz constant. Then at any x, we have

$$|g'(x)| = \lim_{x \to c} \frac{|g(x) - g(c)|}{|x - c|} \leq \lim_{x \to c} M = M.$$

(g) Use the result of part (d) to show that the function $f : (0, 1] \to \mathbf{R}$ is not Lipschitz continuous.

Using the product rule and the chain rule, we have that the derivative of f on $(0, 1)$ is $f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$. As $x \to 0$, this becomes unbounded (the sine and cosine terms are bounded, while $\frac{1}{x}$ blows up). Thus by the previous part, f is not Lipschitz on $(0, 1]$.

3. (a) Define what it means for a function to be Riemann integrable. Include definitions of $U_P(f)$ and $L_P(f)$.

A function f is Riemann integrable on $[a, b]$ if $\inf_P U_P(f) = \sup_P L_P(f)$, where

$$U_P(f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \qquad L_P(f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}),$$

where P is the partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, and

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x) \qquad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x).$$

(b) Prove one of the following two theorems:

Theorem 1. *A continuous function on a closed interval is Riemann integrable.*

Proof. Let $\epsilon > 0$ be given. If f is continuous on the closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$. Thus there is some $\delta > 0$ so that $|f(x) - f(y)| < \epsilon/(b - a)$ whenever $|x - y| < \delta$. Then as long as P is a partition with subintervals no larger than δ, we have

$$|U_P(f) - L_P(f)| = |\sum M_i(x_i - x_{i-1}) - \sum m_i(x_i - x_{i-1})|$$

$$= |\sum (M_i - m_i)(x_i - x_{i-1})| \leq \left| \frac{\epsilon}{b - a} \sum (x_i - x_{i-1}) \right| = \epsilon.$$

□

Theorem 2. *A bounded monotone function on a closed interval is Riemann integrable.*

Proof. Let $\epsilon > 0$ be given. We may assume that f is monotone increasing. Thus we have $M_i = f(x_i)$ and $m_i = f(x_{i-1})$ for all i. Let M be a bound on f, and let P be any partition with $x_i - x_{i-1} < \epsilon/2M$. Then we have

$$|U_P(f) - L_P(f)| = |\sum M_i(x_i - x_{i-1}) - \sum m_i(x_i - x_{i-1})|$$

$$= \left|\sum (f(x_i) - f(x_{i-1}))(x_i - x_{i-1})\right| < (f(b) - f(a))\frac{\epsilon}{2M}$$

$$\leq \frac{\epsilon}{2M} 2M = \epsilon,$$

where we have used the fact that the sum of the $f(x_i) - f(x_{i-1})$ is telescoping. □

(c) Suppose we define a function $f : [0,1] \to \mathbf{R}$ as follows: if x is a rational number, say $x = p/q$ in lowest terms, then we define $f(p/q) = 1/q$; if x is irrational, we define $f(x) = 0$. Show that $L_P(f) = 0$ for any partition P of $[0,1]$.

Let P be a partition. Note that f is always bounded below by zero. Also, every interval $[x_{i-1}, x_i]$, contains some irrational number. The function f is defined to be zero at all irrational numbers. Thus in every interval of the partition, f takes on its minimum value of zero. It follows that $m_i = 0$ for all i. Thus $L_P(f) = 0$ for all partitions P.

(d) Suppose we can show that for any $\epsilon > 0$, there is a partition P of $[0,1]$ so that $U_P(f) < \epsilon$. State carefully the theorem that allows you to deduce that f is Riemann integrable on $[0,1]$.

If, for any $\epsilon > 0$, there is a partition P so that $|U_P(f) - L_P(f)| < \epsilon$, then f is Riemann integrable. (Note that because $L_P(f) = 0$ for all P, we have in this case that $|U_P(f) - L_P(f)| = |U_P(f)|$.)

(e) Assuming this function f is Riemann integrable on $[0,1]$, what is the value of $\int_0^1 f(x)\,dx$?

Because $L_P(f) = 0$ for all P, we must have $\int_0^1 f(x)\,dx = 0$.

(f) Show that $U_{P_n}(f) \leq \epsilon$, where $P_n : 0 = x_0 < x_1 < x_2 < \cdots < x_{2n+1} = 1$ is a partition of $[0,1]$ with $2n+1$ subintervals having the following properties:

on the even subintervals (of the form $[x_{\text{odd}}, x_{\text{even}}]$) the function f is bounded above by $\epsilon/2$;

the odd subintervals (of the form $[x_{\text{even}}, x_{\text{odd}}]$) all have length no more than $\frac{\epsilon}{2(n+1)}$.

We split the upper sum into two parts, the evens and the odds. This gives

$$U_{P_n}(f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i \text{ odd}} M_i(x_i - x_{i-1}) + \sum_{i \text{ even}} M_i(x_i - x_{i-1}).$$

By assumption, $M_i \leq \epsilon/2$ when i is odd, and $(x_i - x_{i-1}) \leq \frac{\epsilon}{2(n+1)}$ when i is even. Also, for all i we have $M_i \leq 1$. This gives us

$$U_{P_n}(f) \leq \sum_{i \text{ odd}} \frac{\epsilon}{2}(x_i - x_{i-1}) + \sum_{i \text{ even}} \frac{\epsilon}{2(n+1)}$$

$$= \frac{\epsilon}{2} \sum_{i \text{ odd}} (x_i - x_{i-1}) + \frac{\epsilon}{2(n+1)} \sum_{i \text{ even}} 1 \leq \frac{\epsilon}{2} + \frac{\epsilon}{2(n+1)}(n+1) = \epsilon.$$