Test 1 Solutions Calculus I September 12, 2008

1. The graph of a function $y = f(x)$ is shown below. Determine the following limits or state that the limit does not exist:

A) $\lim_{x\to 0^+} f(x) = 0$

B) $\lim_{x\to 5^-} f(x) = 2$

C) $\lim_{x\to 5^+} f(x) = 1$

D) $\lim_{x\to 5} f(x)$ DNE

E) $\lim_{x\to 6^-} f(x) = \infty$

F) $\lim_{x\to 6^+} f(x) = \infty$

G) $\lim_{x\to 6} f(x)$ DNE

H) $\lim_{x\to 7^-} f(x) = -\infty$

I) $\lim_{x\to 7^+} f(x) = 2$

J) $\lim_{x\to 7} f(x)$ DNE

2. Evaluate the limits using the limit laws, along with the facts that $\lim_{x\to a} k = k$ and $\lim_{x\to a} x = a$:

A) $\lim_{x\to -1} (x^3 - 2x^2 + 1) = (\lim_{x\to -1} x)^3 - 2(\lim_{x\to -1} x)^2 + \lim_{x\to -1} 1 = -1 - 2 + 1 = -2.$

B) $\lim_{t\to 1} \dfrac{t^2 - t}{t+1} = \dfrac{(\lim_{t\to 1} t)^2 - \lim_{t\to 1} t}{\lim_{t\to 1} t + \lim_{t\to 1} 1} = \dfrac{1-1}{1+1} = 0.$

C) $\lim_{x\to 0} \dfrac{1+\cos x}{x^3+2} = \dfrac{\lim_{x\to 0} 1 + \cos(\lim_{x\to 0} x)}{(\lim_{x\to 0} x)^3 + \lim_{x\to 0} 2} = \dfrac{1+1}{0+2} = 1.$

3. Calculate the following, given that $\lim_{x\to 3} f(x) = 2$ and $\lim_{x\to 3} g(x) = 5$:

A) $\lim_{x\to 3} \dfrac{3f(x)}{g(x) - x} = \dfrac{3(2)}{5-3} = 3.$

B) $\lim_{x\to 3} x^2 f(x) = 3^2(2) = 18.$

4. For what value(s) of c is the following function continuous:

$$f(x) = \begin{cases} x+1 & x < 2 \\ c - x^2 & x \geq 2 \end{cases}$$

We set the expressions equal with $x = 2$, obtaining $2 + 1 = c - 2^2$, so that $c = 7$.

5. Evaluate the limit or state that it does not exist:

A) $\lim_{x\to 1} \dfrac{x^3 - x}{x - 1} = \lim_{x\to 1} \dfrac{x(x+1)(x-1)}{x-1} = \lim_{x\to 1} \dfrac{x(x+1)}{1} = 2.$

B) $\lim_{x\to 0} \dfrac{\sin^2(x)}{x} = \lim_{x\to 0} \dfrac{\sin(x)}{x} \sin(x) = (1) \lim_{x\to 0} \sin(x) = 0.$

C) $\lim_{x\to \pi/2} \dfrac{\cos(x) - 1}{\sin(x)} = \dfrac{0-1}{1} = -1.$

D) $\lim_{x\to 0} x^2 \sin\left(\dfrac{1}{x}\right)$

We use the squeeze theorem. Beginning with $-1 \leq \sin(1/x) \leq 1$, we multiply by x^2 to obtain $-x^2 \leq x^2 \sin(1/x) \leq x^2$. Letting $x \to 0$, the outermost expression both limit to zero. The squeeze theorem then implies that the inner function also limits to zero as $x \to 0$.

7. Use the IVT to find a interval of length one in which lies an x so that $e^{-x^2} = x$.

Note that $e^{-x^2} - x$ is continuous for all x. Also, plugging in $x = 0$ gives $e^0 - 0 = 1 - 0 = 1 > 0$, while plugging in $x = 1$ gives $e^{-1} - 1 = \frac{1}{e} - 1 < 0$. It follows form the intermediate value theorem that $e^{-x^2} = x$ for some x between zero and one.