Test 3 Solutions Calculus I November 10, 2008

1. (a) Use Rolle's theorem to show that $g(x) = x^5 + 2x^3 + 4x - 12$ has at most one real root.

> The derivative is $g'(x) = 5x^4+6x^2+4$, which is always positive. Rolle's theorem tells us that between any two roots of g, the derivative must be zero somewhere. As the derivative never is zero, there must not be two roots. Thus there is at most one.

(b) Suppose that $f(0) = 4$ and that $f'(x) \leq 2$ for all x. Show that $f(3) \leq 10$.
Hint: Use the Mean Value Theorem inequality $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

> Plugging $a = 0$ and $b = 3$ into the inequality, we find
> $$f'(c) = \frac{f(3) - f(0)}{3 - 0} = \frac{f(3) - 4}{3},$$
> which implies that
> $$f(3) \leq 3f'(c) + 4.$$
> This is true for some c with $0 < c < 3$. But we know that $f'(x) \leq 2$ for all x (in particular, for $x = c$). Thus we have
> $$f(3) \leq 3f'(c) + 4 \leq 3(2) + 4 = 10.$$

2. Consider the function $f(x) = x^4 - 6x^2 + 5$.

(a) Find the critical points for this function and indicate the intervals on which the function is increasing or decreasing.

> We compute
> $$f'(x) = 4x^3 - 12x = 4x(x^2 - 3) = 0 \qquad \Rightarrow \qquad x = 0, \quad x = \pm\sqrt{3}.$$
> These are the three critical points. Plugging in intermediate values, we find
> $$f'(-2) = -8 < 0 \qquad f'(-1) = 8 > 0 \qquad f'(1) = -8 < 0 \qquad f'(2) = 8 > 0,$$
> from which we deduce that the function is increasing on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$ and decreasing on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$.

(b) Find the inflection points for this function and indicate the intervals on which the function is concave up or concave down.

> We compute
> $$f''(x) = 12x^2 - 12 = 12(x + 1)(x - 1) = 0 \qquad \Rightarrow \qquad x = \pm 1.$$
> These are the two inflection points. Plugging in intermediate values, we find
> $$f''(-2) = 36 > 0 \qquad f''(0) = -12 < 0 \qquad f''(2) = 36 > 0,$$
> from which we deduce that the function is concave up on $(-\infty, -1)$ and $(1, \infty)$ and concave down on $(-1, 1)$.

3. Compute the following limit: $\lim_{x\to 0} \dfrac{e^x - x - 1}{\cos x - 1}$.

Plugging in $x = 0$, we obtain $\frac{1-0-1}{1-1} = \frac{0}{0}$. This is an indeterminant form, so we may apply L'Hopital's Rule:

$$\lim_{x\to 0} \frac{e^x - x - 1}{\cos x - 1} = \lim_{x\to 0} \frac{e^x - 1}{-\sin x}.$$

Plugging in $x = 0$ to this also gives $\frac{0}{0}$, so we use L'Hopital's again:

$$\lim_{x\to 0} \frac{e^x - 1}{-\sin x} = \lim_{x\to 0} \frac{e^x}{-\cos x}.$$

Plugging in to this gives -1 as the limit.

4. Suppose $f''(x) = 24x^2 + 6x$, with $f'(1) = 8$ and $f(1) = 1$. Find $f(x)$.

Antidifferentiating gives $f'(x) = 8x^3 + 3x^2 + C$. Using the fact that $f'(1) = 8$, we find that $C = -3$, so that $f'(x) = 8x^3 + 3x^2 - 3$. Antidifferentiating again gives $f(x) = 2x^4 + x^3 - 3x + D$. Using $f(1) = 1$, we find that $D = 1$, so that

$$f(x) = 2x^4 + x^3 - 3x + 1.$$

5. An 8-billion-bushel corn crop brings a price of \$2.40 per bushel. A commodity broker uses the following rule of thumb: If the crop is reduced by x percent, then the price increases by $10x$ cents. Which crop size results in maximum revenue and what is the price per bushel?

We're maximizing revenue, which is

$$R = (\text{number of bushels})(\text{price per bushel}).$$

Decreasing an 8-billion bushel crop by $x\%$ leaves

$$8,000,000,000\left(1 - \frac{x}{100}\right) = \text{number of bushels harvested.}$$

The price corresponding to this decrease is $240 + 10x$. Thus the revenue when the crop size is decreased by $x\%$ is

$$R(x) = 8,000,000,000\left(1 - \frac{x}{100}\right)(240 + 10x)$$
$$= 8,000,000,000\left(240 + 10x - 2.4x - \frac{x^2}{10}\right).$$

Taking derivatives gives

$$R'(x) = 8,000,000,000\left(7.6 - \frac{x}{5}\right).$$

Setting this equal to zero and solving, we have

$$x = 5(7.6) = 38.$$

Thus we should decrease the crop size by 38% and charge $\$2.40 + \$3.80 = \$6.20$ per bushel.