Topology I Final Exam Name:

Due in my mailbox or office Monday, Dec 8, by 4 pm. Feel free to use your books (Armstrong and Hatcher) and notes, but please do not use any other resources.

1. Let X be the real line with the half-open interval topology, and let $Y = X \times X$ be given the product topology.

(a) Show that $\mathcal{B} = \{(a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}$ is a base for the standard topology on $\mathbf{R}$.

(b) Recall that X has base $\mathcal{B}_{1/2} = \{\,[a, b) \mid a < b\}$. Show that the base $\mathcal{B}' = \{\,[a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}$ generates a topology different from the half-open interval topology. (Hint: Find a set that is open in the half-open interval topology that cannot be expressed appropriately in terms of sets in $\mathcal{B}'$.)

(c) Show that Y is separable.

(d) Show that the line $y = 1 - x$ is a non-separable subspace of Y. (Hint: What is the subspace topology on this line?)

(e) Show that X is Lindelöf but Y is not. (Hint: The Lindelöf property is inherited by closed subspaces. Show that the line of part (d) is closed and not Lindelöf.)

2. Let X be the set of real numbers with the finite complement topology (complements of finite sets are open).

(a) Is X Hausdorff? Why or why not?

(b) To what point or points does the sequence $x_n = 1/n$ converge?

3. Let $\mathbf{R}^\omega$ denote the set of all sequences $(a_1, a_2, a_3, \ldots)$ of real numbers. We will think of $\mathbf{R}^\omega$ as a product $\mathbf{R} \times \mathbf{R} \times \cdots$ (with infinitely many factors), but topologically things are somewhat subtle. Define the *box topology* on $\mathbf{R}^\omega$ to be the topology generated by sets of the form $U_1 \times U_2 \times \cdots$ where U_n is open in $\mathbf{R}$. Define the *product topology* to be generated by sets of this same form except that all but finitely many of the U_n are required to be all of $\mathbf{R}$. Define a map $f : \mathbf{R} \to \mathbf{R}^\omega$ by $f(t) = (t, \frac{1}{2}t, \frac{1}{4}t, \frac{1}{8}t, \ldots)$. Let $A \subset \mathbf{R}^\omega$ be defined by

$$A = \{(x_n) \in \mathbf{R}^\omega \mid x_n = 0 \text{ for all but finitely many } n\}.$$

(a) Show that f is continuous when $\mathbf{R}^\omega$ is given the product topology.

(b) Show that f is not continuous when $\mathbf{R}^\omega$ is given the box topology.

(c) Prove that A is dense in $\mathbf{R}^\omega$ with the product topology.

(d) Let $B \subset \mathbf{R}^\omega$ be the set of all bounded sequences. Prove that B is both open and closed in $\mathbf{R}^\omega$ with the box topology.

(e) Conclude that A is not dense in $\mathbf{R}^\omega$ with the box topology.

4. Consider the following subspace D of the plane. Let $K = \{1/n \mid n \in \mathbf{Z}_+\}$ and define

$$D = ([0,1] \times 0) \cup (K \times [0,1]) \cup (0 \times 1).$$

Let $p \in D$ be the point 0×1.

(a) Is D connected? Why or why not?

(b) Show that D is not path-connected, as follows. Consider any path $f : [0,1] \to D$ with $f(0) = p$. Show that $f(1)$ must also by p, by showing that $f^{-1}(\{p\}) = [0,1]$. (Hint: Show that $f^{-1}(\{p\})$ is open in $[0,1]$. Then show that it is closed.)

5. Let $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected and Y is connected, then X is connected.

6. Let A denote the unit square $[0,1] \times [0,1]$ under the equivalence relation $(0,t) \sim (1,t)$ for $0 \leq t \leq 1$. Let M denote the unit square under the equivalence relation $(0,t) \sim (1, 1-t)$ for $0 \leq t \leq 1$. A *retraction* is a map $f : X \to E$ where $E \subset X$ so that $f(e) = e$ for all $e \in E$.

(a) Show that A is homeomorphic to the product $S^1 \times [0,1]$.

(b) Show that B is not a topological product.

(c) Show that if there is a retraction $f : X \to E$, then X and E are homotopy equivalent.

(d) Construct retractions $r_1 : A \to S^1$ and $r_2 : M \to S^1$, where S^1 is the circle $\{(t, \frac{1}{2}) \mid 0 \leq t \leq 1\}/\sim$ (where $\sim$ is the appropriate equivalence relation).

7. The purpose of this problem is to show that the continuity of functions *into* product spaces can be deduced one coordinate at a time, but no such similar result holds for functions *from* product spaces.

(a) Show that a function $f : Y \to X$ from a space Y into a product space $X = X_1 \times X_2$ is continuous if and only if the compositions $p_1 f$ and $p_2 f$ are continuous, where p_i is the projection map.

(b) Consider the following function $f : \mathbf{R}^2 \to \mathbf{R}$:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that f is continuous in the first variable and in the second variable, but is not continuous at $(0,0)$. (A function $f : X_1 \times X_2 \to Y$ is *continuous in the first variable* if for each $z \in X_2$ the map $f_1 : X_1 \to Y$ defined by $f_1(x) = f(x,z)$ is continuous. Continuity in the second variable is similarly defined.)