Homework 1 Solutions

I.1: Prove that $v(T) - e(T) = 1$ for any tree T.

We will proceed by induction on the number e of edges in the tree. If $e(T) = 1$, then we know exactly what T looks like, and we find that $v(T) - e(T) = 2 - 1 = 1$, as required.

Now suppose we know the result for any tree with no more than n edges, and let T be a tree with $n+1$ edges. Let T' be a subtree of T with n edges*. Thus $v(T') - e(T') = 1$. Now clearly we have $e(T) = e(T') + 1$. On the other hand, the edge in $T - T'$ must share at least one vertex with T' (because otherwise T would not be connected), but cannot share both with it (because otherwise there would be a loop formed by the new edge and any path in T' joining the two vertices to which the ends of the new edge are attached (such a path exists because T' is connected)). Thus $v(T) = v(T') + 1$. The result follows.

*(The existence of such a subtree requires some justification. In particular, we can't just remove any old edge, for this may disconnect T, so that T' is not a tree, but rather a disjoint union of two trees. To show that it exists, we need to show that there is at least one edge in T that has a 'free' vertex; i.e., a vertex not contained in any other edge. To find it, pick any edge e_1. If it has a free vertex, you're done. If not, move to an adjacent edge e_2. Repeat, being sure never to reverse direction. There are only $n+1$ edges, so at some point either this process terminates, or you repeat an edge. If you repeat an edge, however, you will have traversed a loop, contradicting that T is a tree. Thus the process terminates at an edge with a free vertex.)

I.3: Show that inside any graph we can always find a tree which contains all the vertices.

If the graph is already a tree, then we are finished. We claim that if the graph is not a tree, then there is some edge in it that can be removed without disconnecting the graph. To see this, note that if the graph is not a tree, then there must be a loop in it. Remove any edge in the loop. The remaining portion of the loop provides a path joining the two vertices that the removed edge previously joined. Also note that all the vertices of the old graph are still vertices of the new graph, again because each endpoint of the removed edge must have been contained in some other edge also (because the removed edge was in a loop).

When we remove such a non-separating edge, as long as the resulting graph is not a tree, we can repeat the process. On the other hand, the process cannot possibly be repeated forever, because the graph has finitely many edges, and the only graph with one edge is a tree. Thus the process terminates at some tree.

I.6: Let P be a regular polyhedron in which each face has p edges and for which q faces meet at each vertex. Using Euler's formula prove that

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{2}+\frac{1}{e}.$$

The fact that exactly two polygons meet along a given edge shows that $pf = 2e$ (the quantity pf counts p edges for each face, resulting in each edge being counted twice). Because q faces meet at a vertex, there are q edges meeting at any given vertex. Then the fact that each edge has exctly two vertices implies that $qv = 2e$ (the quantity qv counts q edges for each vertex, resulting in each edge being counted twice). Substituting these equations into Euler's formula yields

$$2 = v - e + f = \frac{2e}{q} - e + \frac{2e}{p},$$

which simplifies to the given equation.

I.7: Deduce from Problem 6 that there are only five regular polyhedra.

First note that we must have $p, q \geq 3$. (That $p \geq 3$ follows from the fact that the fewest edges a polygon can have is three, in the triangle; that $q \geq 3$ is a bit more subtle, but not geometrically difficult.) Once this is established, we need only go through the options, noting that the formula from I.6 shows that $\frac{1}{p}+\frac{1}{q} > \frac{1}{2}$. If $p = 3$, then the only values of q that satisfy this inequality are $q = 3, 4, 5$. If $p = 4$, then the only value of q that satisfies this inequality is $q = 3$. If $p = 5$, then the only value of q that satisfies this inequality is $q = 3$. Thus there are at most five regular polyhedra, corresponding to (p, q) values of $(3, 3)$, $(3, 4)$, $(3, 5)$, $(4, 3)$, and $(5, 3)$ (these are the tetrahedron, octahedron, icosahedron, cube, and dodecahedron, respectively).

I.10: Find a homeomorphism from the real line to the open interval $(0, 1)$. Show that any two open intervals are homeomorphic.

A standard choice is $f(x) = \tan^{-1}(x)$, which maps $\mathbf{R}$ continuously to the open interval $(-\pi/2, \pi/2)$ and has continuous inverse. We then need to map this interval to $(0, 1)$. We can do this ''linearly'' by sending endpoints to endpoints, using the map $g(x) = \frac{x}{\pi} + \frac{1}{2}$. This map is also continuous with continuous inverse. Thus their composition is a homeomorphism.

The map $f(x) = ax+b$ sends the interval (p, q) homeomorphically to the interval (r, s), where $a = \frac{r-s}{p-q}$ and $b = \frac{sp-rq}{p-q}$.

I.12: 'Stereographic projection' π from the sphere minus the north pole N to the plane is defined geometrically as follows: for a point x on the sphere we define $\pi(x)$ to be the unique point where the line through N and x intersects the xy-plane. Work out a formula for π and check that π is a homeomorphism. Notice that π provides us with a homeomorphism from the sphere with the north and south poles removed to the plane minus the origin.

Suppose x has coordinates (a, b, c). Then the line through $N = (0, 0, 1)$ and x has parametrization $\mathbf{r(t)} = \langle \mathbf{ta}, \mathbf{tb}, \mathbf{1{+}t(c{-}1)} \rangle$. This line hits the xy-plane when $1 + t(c-1) = 0$, or when $t = \frac{1}{1-c}$. Plugging in this value of t, we find the x- and y-coordinates of the intersection point to be $\pi(a, b, c) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right)$. This is clearly continuous when $c \neq 1$, and it is not too hard to see that it has a continuous inverse.