Homework 2 Solutions

I.19: Let X be a topological space and let Y be a subset of X. Check that the so-called subspace topology is indeed a topology on Y.

In the subspace topology, a subset B of Y is open precisely if it can be written as $B = A \cap Y$ for some set A open in X. To check that this yields a topology on Y, we check that it satisfies the axioms.

(i) $\emptyset$ an Y are open in Y: To verify this we write $\emptyset = \emptyset \cap Y$ and $Y = X \cap Y$; the claim then follows because $\emptyset$ and X are open in X.

(ii) Arbitrary unions of sets open in Y are open in Y: For this, suppose we have an arbitrary collection B_λ of open sets in Y. Then for each B_λ we may write $B_\lambda = A_\lambda \cap Y$ where A_λ is some open set in A. We know that $A = \bigcup A_\lambda$ is open in X. Then (using deMorgan's laws) we have

$$\bigcup B_\lambda = \bigcup (A_\lambda \cap Y) = (\bigcup A_\lambda) \cap Y = A \cap Y.$$

(iii) Finite intersections of sets open in Y are open in Y: Given a collection of open sets $B_1, \ldots, B_n$ in Y, we write $B_j = A_j \cap Y$ for A_j open in X. We know that $A = \bigcap A_j$ is open in X. Then

$$\bigcap B_j = \bigcap (A_j \cap Y) = (\bigcap A_j) \cap Y = A \cap Y.$$

II.1: Verify each of the following for arbitrary subsets A and B of a space X:

(a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

To prove equality, we show containment in both directions. To show that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$, we choose a point $x \in \overline{A \cup B}$. Then any open set $\mathcal{O}_x$ containing x has $\mathcal{O}_x \cap (A \cup B) \neq \emptyset$. But then $(\mathcal{O}_x \cap A) \cup (\mathcal{O}_x \cap B) \neq \emptyset$. It follows that either $x \in \overline{A}$ or $x \in \overline{B}$. In either case, we have that $x \in \overline{A} \cup \overline{B}$.

Now we need to show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. For this, suppose that $x \in \overline{A} \cup \overline{B}$. Then any open set $\mathcal{O}_x$ containing x intersects either A or B nontrivially. In particular, $\mathcal{O}_x \cap (A \cup B) \neq \emptyset$. The result follows.

(b) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Show that equality need not hold.

Choose a point $x \in \overline{A \cap B}$. Then any open set $\mathcal{O}_x$ containing x intersects $A \cap B$ nontrivially. But this means that $\mathcal{O}_x$ intersects both A and B nontrivially. Thus $x \in \overline{A}$ and $x \in \overline{B}$, so that $x \in \overline{A} \cap \overline{B}$.

To see that equality need not hold, consider $A = \{x \in \mathbf{R} \mid x > 0\}$ and $B = \{x \in \mathbf{R} \mid x < 0\}$. Then $A \cap B = \emptyset$, so that $\overline{A \cap B} = \emptyset$. On the other hand, $\overline{A} = \{x \in \mathbf{R} \mid x \geq 0\}$ and $\overline{B} = \{x \in \mathbf{R} \mid x \leq 0\}$, so that $\overline{A} \cap \overline{B} = \{0\}$.

(e) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

We first show that $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$. For this, take a point $x \in (A \cap B)^\circ$. This means that $x \in \mathcal{O}_x$ for some open set $\mathcal{O}_x$ with $\mathcal{O}_x \subseteq A \cap B$. But this means that $\mathcal{O}_x \subseteq A$ and $\mathcal{O}_x \subseteq B$, so that $x \in A^\circ$ and $x \in B^\circ$. Thus $x \in A^\circ \cap B^\circ$.

For the other containment, suppose $x \in A^\circ \cap B^\circ$. Then we have $x \in \mathcal{O}_x \subseteq A$ and $x \in \mathcal{P}_x \subseteq B$ for some open sets $\mathcal{O}_x$ and $\mathcal{P}_x$. Then $\mathcal{Q}_x = \mathcal{O}_x \cap \mathcal{P}_x$ is an open set containing x and contained in $A \cap B$, so that $x \in (A \cap B)^\circ$.

II.3: Specify the interior, closure and frontier of the following subsets of the plane:

(a) $\{(x, y) \mid 1 < x^2 + y^2 \leq 2\}$.

The interior is the open annulus $\{(x, y) \mid 1 < x^2 + y^2 < 2\}$. The closure is the closed annulus $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 2\}$. The frontier is the disjoint of the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$.

(b) $\mathbb{E}^2$ with both axes removed.

The interior is the set itself. The closure is all of $\mathbb{E}^2$. The frontier is the two axes.

(c) $\mathbb{E}^2 - \{(x, \sin(1/x) \mid x > 0\}$.

The interior is the set itself minus the piece of y-axis between $y = -1$ and $y = 1$. The closure is all of $\mathbb{E}^2$. The frontier is the graph of $y = \sin(1/x)$ for $x > 0$, along with the piece of y-axis mentioned above.