Homework 3 Solutions

II.12: Show that if X has a countable base for its topology, then X contains a countable dense subset. (A space whose topology has a countable base is called a *second countable* space. A space that contains a countable dense subset is said to be *separable*.)

> Let $\mathcal{B} = \{B_1, B_2, \ldots\}$ be a countable base for the topology on X. For each $n \in \mathbf{N}$, choose a point $x_n \in B_n$. We claim that $D = \{x_1, x_2, \ldots\}$ is dense in X. To see this, choose any point $x \in X \setminus D$. We need to show that any open set containing x also contains some point in D. Suppose $\mathcal{O}_x$ is an open set containing x. Then by the definition of a base, there is some k for which $B_k \subseteq \mathcal{O}_x$. But then $x_k \in \mathcal{O}_x$.

II.13: If $f\colon \mathbf{R} \to \mathbf{R}$ is a continuous function, show that the set of points that are left fixed by f is a closed subset of $\mathbf{R}$. If g is a continuous real-valued function on X show that the set $\{x \mid g(x) = 0\}$ is closed.

> We prove the second claim first. The set $\{0\} \subset \mathbf{R}$ is closed. Because f is continuous, the preimage of a closed set is closed. Thus $g^{-1}(0) = \{x \mid g(x) = 0\}$ is closed.
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> For the first claim, we simply define a function $g\colon \mathbf{R} \to \mathbf{R}$ by $g(x) = f(x) - x$. Then the set of points fixed by f is precisely the set $\{x \in \mathbf{R} \mid f(x) = x\} = \{x \in \mathbf{R} \mid f(x) - x = 0\} = \{x \in \mathbf{R} \mid g(x) = 0\}$, which is closed by the second claim.

II.16: What topology must X have if every real-valued function defined on X is continuous?

> It must be that X has the discrete topology. For given any set $A \subseteq X$, I can define a function $F\colon X \to \mathbf{R}$ that sends points in A to zero and all other points to one. If this function is to be continuous, then (by the previous problem), A must be closed. Because A is arbitrary, this means that every subset of X is closed. It follows that every subset of X is open. This is the discrete topology.

II.17: Let X denote the set of all real numbers with the finite-complement topology, and define $f\colon \mathbb{E}^1 \to X$ by $f(x) = x$. Show that f is continuous, but it is not a homeomorphism.

Let A be an open set in X. Thus $\mathbf{R} \backslash A$ is finite. Then $f^{-1}(A) = A \subset \mathbb{E}^1$. Note that all finite subsets of $\mathbb{E}^1$ are closed (they are finite unions of closed (single-point) sets). Thus A is open in $\mathbb{E}^1$.

To see that the inverse of f is not continuous, note that $I = (0,1)$ is open in $\mathbb{E}^1$, but it is not open in X. Thus $f^{-1}(I) = I \subset X$ is not open, so f^{-1} is not continuous.