Homework 4 Solutions

III.1: Find an open cover of $\mathbb{E}^1$ that does not contain a finite subcover. Do the same for $[0,1)$ and $(0,1)$.

For the first, consider the cover $\{(\frac{n}{2}, \frac{n}{2}+1)\}_{n=-\infty}^{\infty}$. Then for every n, the number $\frac{n}{2}+\frac{1}{2}$ is contained in only one of these sets, namely the interval $(\frac{n}{2}, \frac{n}{2}+1)$. Thus this cover has no subcovers at all, let alone any finite ones.

For the second, take the open cover $\{[0, 1-\frac{1}{n})\}_{n=2}^{\infty}$. (Note that, despite appearances, these sets are open in the subspace topology on $[0,1)$.) While any finite subcollection of these sets may be discarded (as, in fact, can some infinite subcollections), we must always retain an infinite number of them in order to cover all of $[0,1)$. Thus there is no finite subcover, so the interval is not compact.

For $(0,1)$, we take the cover $\{(0, 1-\frac{1}{n})\}_{n=2}^{\infty}$. The comments about $[0,1)$ also apply here.

III.3: Use the Heine-Borel theorem to show that an infinite subset of a closed bounded interval must have a limit point.

Let X be an infinite subset of the closed interval $[a,b]$, and suppose for contradiction that X has no limit points. Then every point $y \in [a,b]$ is contained in some open set $\mathcal{O}_y$ so that either $\mathcal{O}_y \cap X = \emptyset$ or $\mathcal{O}_y \cap X = y$.

These $\mathcal{O}_y$ form an open cover of $[a,b]$. Now because $[a,b]$ is closed and bounded, the Heine-Borel theorem implies that $[a,b]$ is compact. Thus this open cover has some finite subcover $\{\mathcal{O}_z\}$, where the z index ranges over some *finite* subset Z of $[a,b]$.

Because Z is finite, while X is infinite, there must be some $x \in X$ with $x \notin Z$. Thus there is some $z \neq x$ so that $x \in \mathcal{O}_z$. But this contradicts the choice of the open set $\mathcal{O}_z$, which was supposed to either be disjoint from X or intersect it only in z itself. This contradiction shows that the assumption that X has no limit point is fallacious.