Homework 5 Solutions

III.8 (Lindelöf's theorem): If X has a countable base for its topology, prove that any open cover of X contains a countable subcover.

Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X. For each A_λ choose a base open set B_λ so that $B_\lambda \subseteq A_\lambda$ (we can do this by the definition of a base). Then the B_λ cover X. Moreover, there are only countably many B_λ, so in fact we have a countable open cover $\{B_i\}_{i\in\mathbf{N}}$. Now for each i choose *one* A_i so that $B_i \subseteq A_i$. Then the A_i form a countable subcover of X.

III.11: Find a topological space and a compact subset whose closure is not compact.

Here is one example. Take X to be the disjoint union $\mathbf{R}\cup\{a\}$, where a set is open if and only if it contains the point a (or is empty). Then the single-point subset $\{a\}$ is compact (any one non-empty set in an open cover is a finite subcover), but it is not closed, as its complement is not open (it is neither empty nor contains a). The closure of $\{a\}$ is the smallest closed set containing a, so its complement is the largest open set not containing $\{a\}$. The only such is the empty set; thus the closure of $\{a\}$ is all of X. That X is not compact can be seen by taking as an open cover the collection of all sets of the form $\{a\cup r\}_{r\in\mathbf{R}}$.

III.17: Let X be a locally compact Hausdorff space which is not compact. Form a new space by adding one extra point, usually denoted ∞, to X and taking the open sets of $X \cup \{\infty\}$ to be those of X together with sets of the form $(X - K) \cup \{\infty\}$, where K is a compact subset of X. Check the axioms for a topology, and show that $X \cup \{\infty\}$ is a compact Hausdorff space which contains X as a dense subset. The space $X \cup \{\infty\}$ is called the *one-point compactification* of X.

We check the axioms first. For convenience we denote $X\cup\{\infty\}$ by $\widehat{X}$. Note that $\widehat{X} = (X-\emptyset)\cup\{\infty\}$. The fact that $\emptyset$ is compact shows that $\widehat{X}$ is open. Also, $\emptyset$ is open because it is open in the topology on X.

Consider an arbitrary union of open sets in $\widehat{X}$. Let A be the union of all of the sets in this union that are subsets of X. Then A is open in X, and hence in $\widehat{X}$. The arbitrary union then looks like $\bigcup(X-K_\lambda)\cup\{\infty\}\cup A$. This can be rewritten as $(X-\cap(K_\lambda \cap A^c)) \cup \infty$. It now suffices to show that $\cap(K_\lambda \cap A^c)$ is compact. (By A^c, we mean the complement of A.) But A^c is closed (since A is open), as is each K_λ (because X is Hausdorff). Thus each $K_\lambda \cap A$ is closed. Hence the intersection $\cap(K_\lambda \cap A)$ is closed. As a closed subset of any of the compact K_λ, this intersection is also compact. Thus our arbitrary union of open sets is open.

For finite intersections, we proceed in a similar manner.

To see that $\widehat{X}$ is Hausdorff, suppose a and b are two points in $\widehat{X}$. If both are in X, then they can be separated in X, hence in $\widehat{X}$. Otherwise, we may assume that $b = \infty$. In this case, we use the fact that X is locally compact to choose a compact neighborhood K of a. Then if $\mathcal{O}_a$ is any open set containing a inside K, we have separating sets $\mathcal{O}_a$ and $(X - K) \cup \infty$.

To see that $\widehat{X}$ is compact, note that any open cover of $\widehat{X}$ must include a set of the form $(X-K) \cup \{\infty\}$. In the finite subcover, we include one such set, along with a finite subcover for K.

To see that X is dense in $\widehat{X}$, first note that X is not closed. For if it were, then $\{\infty\}$ would be open, but $\infty = (X - X) \cup \{\infty\}$, and X is not compact. So $\{\infty\}$ is not open, so X is not closed. Thus its closure is strictly larger than itself. The only option is that the closure of X is $\widehat{X}$. Thus X is dense in $\widehat{X}$.