Homework 6 Solutions

III.30: Let X be the set of all points in the plane which have at least one rational coordinate. Show that X, with the induced topology, is a connected space.

There are several ways to do this. We will show that X is path-connected and hence connected. We do this by explicitly constructing paths. Suppose (a,p) and (q,b) are points in X with p,q rational. Then we join these points by the horizontal line running from (a,p) to (q,p) joined to the vertical path from (q,p) to (q,b). Note every point on this combined path has at least one rational coordinate (either p or q). To join points of the form (p,a) and (q,b), we join the vertical line joining (p,a) to $(p,0)$, the horizontal line joining $(p,0)$ to $(q,0)$, and the vertical line joining $(q,0)$ to (q,b). The case of (a,p) and (b,q) is similar.

III.31: Give the set of real numbers the finite-complement topology. What are the components of the resulting space? Answer the same question for the half-open interval topology.

Let X be the real numbers with the finite-complement topology. Then any two non-empty open sets of X intersect nontrivially. Hence X is connected.

Now let X be the real numbers with the half-open interval topology, and let A be any subset of X containing more than one point. Then there are two points a and b in A with $a > b$. Then we may write $A = (A \cap [a, \infty)) \cup (A \cap (-\infty, a))$. This is a disjoint union of open sets. Hence any set with more than one point is not connected. It follows that the connected components are single points.

III.33 (Intermediate Value Theorem): If $f\colon [a,b] \to \mathbf{E}^1$ is a map such that $f(a) < 0$ and $f(b) > 0$, use the connectedness of $[a,b]$ to establish the existence of a point c for which $f(c) = 0$.

Note that $[a,b]$ is connected and f is continuous. Thus $f([a,b])$ is a connected subset of $\mathbf{R}$. In particular it is an interval. Moreover, it is an interval containing both positive and negative points. Thus it contains zero. It follows that $f(c) = 0$ for some $a < c < b$.

III.37: Show that the continuous image of a path-connected space is path-connected.

Suppose X is path-connected and $f\colon X \to Y$ is continuous and onto (because we are only looking at the image of f). Pick two points a, b in Y, and let x and y be points in X so that $f(x) = a$ and $f(y) = b$. (Note that these points need not be unique; just pick any points in the preimages.) Because X is path-connected there is a path γ joining x to y. But then $f \circ \gamma$ is a path joining a to b, so that Y is path-connected.

III.44: Prove that a space which is connected and locally path-connected is path-connected.

Suppose X is a connected, locally path-connected space, and pick a point x in X. Let A be the set of points in X to which x may be joined by a path (i.e., A is the path-component of X containing x). We will show that A is both open and closed. The fact that X is connected then implies that A is either empty or all of X, but it's not empty because it contains x. Thus A is all of X, so X is path-connected.

To see that A is open, we need to show that every point of A contains an open neighborhood in A. Suppose $y \in A$, and let γ be a path joining x to y. Let B_y be an open neighborhood of y. Because X is locally path-connected there is an open set $\mathcal{O}_y$ containing y and contained in B_y that is path-connected. Thus x may be joined to any point in $\mathcal{O}_y$ by adjoining γ to the local path starting at y. Thus A is open.

To see that A is closed, we apply the exact same argument to the complement of A. If x cannot be joined to y, then neither can it be joined to points in $\mathcal{O}_y$ (because y and $\mathcal{O}_y$ are in the same path-component). Thus the complement of A is open, so A is closed. The result follows.