Homework 7 Solutions

III.22: Prove that the product of two second-countable spaces is second-countable, and that the product of two separable spaces is separable.

> Suppose X and Y are both second countable. Thus there are countable bases $\mathcal{U}$ and $\mathcal{V}$ for X and Y respectively. Let $\mathcal{B}$ denote the collection of all sets in $X \times Y$ of the form $U \times V$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. We claim that $\mathcal{B}$ is a countable base for the product topology on $X \times Y$. (The fact that it is countable follows from the fact that the product of two countable sets is countable.)
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> To show that it is a base, we need only show that any open set in $X \times Y$ contains a set of this form. But any open set in $X \times Y$ contains a set of the form $A \times B$ for some open sets $A \subset X$ and $B \subset Y$, while A contains a base set U and B contains a base set V. Thus any open set in $X \times Y$ contains a product of base sets $U \times V$, as required.
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> Now suppose that X and Y are both separable. Thus they contain countable dense subsets P and Q. We claim that $P \times Q$ is a countable dense subset of $X \times Y$. (Countability follows as above.) To see that it is dense, let U be an open set in $X \times Y$. We need to show that U intersects $P \times Q$ nontrivially. But the projection of U to X is open in X (and nonempty), and thus intersects P, while the projection to Y intersects Q. It follows that there is some (p, q) in U.

III.25: Show that the diagonal map $\Delta\colon X \to X \times X$ defined by $\Delta(x) = (x, x)$ is indeed a map, and check that X is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$.

> Let U be an open set in $X \times X$. Then U contains a basic open set of the form $U_1 \times U_2$ for some pair of open sets U_1 and U_2 in X. Note that $U \cap \Delta(X) = \{(x, x) \mid x \in U_1 \cap U_2\}$. But then $\Delta^{-1}(U) = U_1 \cap U_2$, which is open in X. Thus Δ is a map.
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> Now suppose that X is Hausdorff and choose any point (x_1, x_2) in $X \times X$. We aim to show that if $x_1 \neq x_2$ (so that $(x_1, x_2) \notin \Delta(X)$), then there is some open set around (x_1, x_2) also not in $\Delta(X)$. This will show that the complement of $\Delta(X)$ is open in $X \times X$, so that $\Delta(X)$ is closed in $X \times X$. Now because X is Hausdorff, there are open sets U_1 and U_2 in X that are disjoint and contain x_1 and x_2, respectively. But then $U_1 \times U_2$ is open in $X \times X$, contains (x_1, x_2), and is disjoint from $\Delta(X)$ (because $U_1 \cap U_2 = \emptyset$).
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> For the converse, suppose $\Delta(X)$ is closed in $X \times X$, and choose $x_1 \neq x_2$ in X. Then there is an open basic set of the form $U_1 \times U_2$ containing (x_1, x_2) in $X \times X$ that is disjoint from $\Delta(X)$. It follows that $x_1 \in U_1$ and $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. Thus X is Hausdorff.

III.39: Prove that the product of two path-connected spaces is path-connected.

> Suppose X and Y are path-connected, and choose two points (x_1, y_1) and (x_2, y_2) in $X \times Y$. Because $X \times \{y_1\}$ is homeomorphic to X, it is path-connected. In particular, there is a path γ_1 in $X \times \{y_1\} \subset X \times Y$ joining (x_1, y_1) to (x_2, y_1). Similarly there is a path γ_2 in $\{x_2\} \times Y \subset X \times Y$ joining (x_2, y_1) to (x_2, y_2). Thus $\gamma_1 \cup \gamma_2$ is a path in $X \times Y$ joining (x_1, y_1) to (x_2, y_2). Thus $X \times Y$ is path-connected.

IV.5: Let X denote the union of the circles $[x-(1/n)]^2+y^2 = (1/n)^2$, $n = 1, 2, 3, \ldots$, with the subspace topology from the plane, and let Y denote the identification space obtained from the real line by identifying all the integers to a single point. Show that X and Y are not homeomorphic. (X is called the *Hawaiian earring.*)

> We claim first that X is compact. To this end, let $\mathcal{U}$ be an open cover of X, and let U_0 be any set in this cover that contains the origin $(0, 0)$. Because we are using the subspace topology, we can write $U_0 = X \cap V$ for some open set V in $\mathbf{R}^2$ containing the origin. Then V must contain a set of the form $\{(x, y) \mid x^2 + y^2 < (1/N)^2\}$ for some N, as sets of this type form a base for the topology on $\mathbf{R}^2$. It follows directly that U_0 contains all circles in X with radius $1/n$ for all $n > N$. In particular, U_0 contains all but finitely many of the circles that make up X. A single circle is compact, and a finite union of compact spaces is compact. Thus there is a finite subset of $\mathcal{U}$ that covers the remaining (finitely many) circles of radius $1/m$ for $m \leq N$. This finite subset, together with U_0, gives a finite subcover for all of X.
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> We claim second that Y is not compact. Consider the cover $\{(n - 1/2, n + 3/2)\}$, for $n \in \mathbf{Z}$, of $\mathbf{R}$. This cover induces a cover of Y having no finite subcover (in fact, no proper subcovers at all).

IV.7: Describe each of the following spaces: (a) the cylinder with each of its boundary circles identified to a point; (b) the torus with the subset consisting of a meridianal and a longitudinal circle identified to a point; (c) S^2 with the equator identified to a point; (d) $\mathbf{E}^2$ with each of the circles centered at the origin and of integer radius identified to a point.

> (a) A 2-sphere; (b) a 2-sphere; (c) two 2-spheres joined at a point; (d) an infinite stack of 2-spheres.