Topology I Test 1 Solutions October 13, 2008

1. Do FIVE of the following:

(a) Give a careful definition of *connected*.

A topological space X is ***connected*** if for any two sets A and B such that $A \cup B = X$, we have either that $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$ (where $\overline{A}$ denotes the closure of A).

(b) Show that X connected and $f : X \to Y$ continuous implies $f(X)$ is connected.

We will use the theorem that gives an equivalent formulation of connected: X is connected if and only if it cannot be written as $X = A \cup B$ where A and B are open, nonempty, and disjoint.

Suppose $f(X)$ is not connected, and choose A and B nonempty and open in $f(X) \subset Y$ so that $A \cup B = f(X)$. Because f is continuous, each of the preimages $f^{-1}(A)$ and $f^{-1}(B)$ is open and nonempty. But because $A \cup B = f(X)$, it is clear that $f^{-1}(A) \cup f^{-1}(B) = X$. This contradicts the assumption that X is connected.

(c) Give a careful definition of *path-connected*.

A topological space X is ***path-connected*** if for any two points $x_0, x_1 \in X$, there is some path (=continuous functions $\gamma : [0,1] \to X$) so that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

(d) Give a careful definition of *closure*.

The ***closure*** of a set A in a topological space X is the union of A along with all of its accumulation points. A point $x \in X$ is an ***accumulation point*** of A if every open set containing x interesects A nontrivially.

(e) Show that if $A \subset X$ is connected, then the closure of A is also connected.

Suppose $\overline{A}$ is not connected, and write $\overline{A} = S \cup T$, where S and T are open, disjoint, and nonempty. Then each of $S \cap A$ and $T \cap A$ is open, and the pair is disjoint. The fact that A is connected means one or the other must be empty. We may assume that $S \cap A$ is empty. Choose $s \in S$. Then S is an open set containing s that is disjoint from A, contradicting the fact that s is a limit point of A.

(f) Show that a path-connected space is connected.

Suppose X is path-connected but not connected, and let $x_0 \in A$ and $x_1 \in B$ be points in distinct connected components of X. (Note then that A is open in X, as is $X \backslash A$.) Let γ be a path joining x_0 to x_1. Then $\gamma^{-1}(A)$ is open in $[0,1]$ as is $\gamma^{-1}(X \setminus A)$. These two sets are open, nonempty, and disjoint, and they cover the interval $[0,1]$. This contradicts the fact that intervals are connected.

(g) Give an example of a connected space that is not path-connected.

Let $X = \{(x, \sin(1/x)) \in \mathbf{R}^2 \mid x > 0\}$ and $Y = \{(0,y) \in \mathbf{R}^2 \mid -1 \leq y \leq 1\}$. Then $X \cup Y$ is connected but not path-connected.

(h) Show that A and B connected and $A \cap B \neq \emptyset$ implies $A \cup B$ is also connected.

Suppose $A \cup B$ is not connected, and write $A \cup B = S \cup T$, where S and T are nonempty disjoint open sets in $A \cup B$. The sets $S \cap A$ and $T \cap A$ are both open in A, and they are disjoint. Because A is connected, one of these must be empty. We may assume that $S \cap A = \emptyset$.

Similarly, we have that $S \cap B$ and $T \cap B$ are open and disjoint in B, and one must be empty. If $S \cap B$ were empty, then $S = \emptyset$, contrary to assumption. Thus we have $T \cap B = \emptyset$.

Now $S \cap A = \emptyset$ implies that $S \subseteq B$. Combined with the facts that $T \cap B = \emptyset$ and $B \subseteq S \cup T$, we deduce that $T = B$. Similarly, we must have $S = A$. But S and T are assumed disjoint, while $A \cap B \neq \emptyset$. This contradiction shows that $A \cup B$ is connected.

2. Do FIVE of the following:

(a) Give a careful definition of *compact*.

A topological space X is *compact* if every open cover of X has a finite subcover.

(b) Give a careful definition of *Hausdorff*.

A topological space is *Hausdorff* if for any distinct two points $a, b \in X$, there are disjoint open sets $\mathcal{O}_a \ni a$ and $\mathcal{O}_b \ni b$.

(c) Show that a closed subset of a compact space is compact.

Suppose X is a compact topological space and that A is a closed subset of X. Let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be an open cover of A. Because A is closed, we know that $X \setminus A$ is open. Thus $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda} \cup (X \setminus A)$ is an open cover of X. Because X is compact, this has a finite subcover, which we may assume is of the form $\{\mathcal{O}_j\}_{j=1}^n \cup (X \setminus A)$. But then $\{\mathcal{O}_j\}_{j=1}^n$ is a finite subcover of A.

(d) Show that a compact subset of a Hausdorff space is closed.

Suppose X is a Hausdorff topological space and that A is a compact subspace. We want to show that A is closed. We will show that the complement $X \backslash A$ of A is open, by showing that every point $x \in X \backslash A$ is contained in an open set $\mathcal{O}_x$ that is disjoint from A.

Fix $x \in X \setminus A$. For each $a \in A$, we may choose open sets $\mathcal{O}_a \ni a$ and $\mathcal{U}_a \ni x$ that are disjoint (this is because X is Hausdorff). Now the union of all the $\mathcal{O}_a$ covers A, so, because A is compact, there is some finite subcover $\{\mathcal{O}_{a_j}\}_{j=1}^n$. Define $\mathcal{O}_x$ to be the intersection of the corresponding $\mathcal{U}_{a_j}$. Then $\mathcal{O}_x$ is open (because it is the finite intersection of open sets), it contains x (because each of the $\mathcal{U}_a$ did), and it is disjoint from A (because it is disjoint from each of the $\mathcal{O}_{a_j}$ which together cover A).

(e) Give an example of a closed set that is not compact.

The real numbers $\mathbf{R}$ are closed but not compact (in the usual topology on $\mathbf{R}$).

(f) Give an example of a compact set that is not closed.

Let X be the union of the real numbers $\mathbf{R}$ and a single point a. Define a topology on X by declaring that a set is open if and only if it is either empty or contains a. Then the set $\{a\}$ is compact (as are all singletons -- given any cover of $\{a\}$, there is a finite subcover consisting of any single one of the original open sets that contained a). It is not closed because its complement is neither empty nor contains a.

(g) Show that if X is compact and $f: X \to Y$ is continuous, then $f(X)$ is compact.

Let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be an open cover of $f(X)$. Because f is continuous, $\{f^{-1}(\mathcal{O}_\lambda)\}_{\lambda \in \Lambda}$ is an open cover of X. Because X is compact, there is a finite subcover, which we may write as $\{f^{-1}(\mathcal{O}_j)\}_{j=1}^n$. Because these sets cover X, the corresponding sets $\{\mathcal{O}_j\}_{j=1}^n$ cover $f(X)$, providing us with our sought-after finite subcover.

3. For each of the following subsets of $\mathbf{R}$, indicate whether the set (in the given topology) is open, closed, both, or neither in the following topologies: (i) usual; (ii) finite complement; (iii) half-open interval.

(a) $[0,1]$	(i) closed	(ii) neither	(iii) closed
(b) $[0,1)$	(i) neither	(ii) neither	(iii) open
(c) $(0,1]$	(i) neither	(ii) neither	(iii) neither
(d) $(0,1)$	(i) open	(ii) neither	(iii) open
(e) $[0,\infty)$	(i) closed	(ii) neither	(iii) open

(f) $(0, \infty)$ (i) open (ii) neither (iii) open

4. Divide the following symbols into homeomorphism classes:

$+ \quad \ominus \quad \pm \quad \rtimes \quad \times \quad \pitchfork \quad \models \quad \in$

A 8 E F H P Y 4

Class 1: $\{+,\ \times,\ \pitchfork\}$

Class 2: $\{\rtimes,\ 4\}$

Class 3: $\{\in,\ \mathsf{E}, \mathsf{F}, \mathsf{Y}\}$

Each of the rest of the symbols lies in its own homeomorphism class.