Linear Algebra I **Test 1 Solutions** **February 7, 2005**

1. Let $A = \begin{bmatrix} -3 & 1 \\ 2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 4 \\ 5 & -4 & -3 \end{bmatrix}$.

If possible, find (a) B^T (b) AB (c) BA (d) A^2 (e) B^2 (e) $A+B$

(a) $B^T = \begin{bmatrix} 2 & 5 \\ -1 & -4 \\ 4 & -3 \end{bmatrix}$ (b) $AB = \begin{bmatrix} -1 & -1 & -15 \\ 39 & -30 & -13 \end{bmatrix}$ (c) BA is not defined

(d) $A^2 = \begin{bmatrix} 11 & 4 \\ 8 & 51 \end{bmatrix}$ (e) B^2 is not defined (f) $A+B$ is not defined

2. Determine for which values of a the following system has 0 solutions, 1 solution, and infinitely many solutions.

$$x + 4y = 2$$

$$3x + (a^2 - 4)y = a + 10$$

The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 4 & | & 2 \\ 3 & a^2 - 4 & | & a + 10 \end{bmatrix},$$

which reduces to

$$\begin{bmatrix} 1 & 4 & | & 2 \\ 0 & a^2 - 16 & | & a + 4 \end{bmatrix}.$$

When $a \neq \pm 4$, we may further reduce to

$$\begin{bmatrix} 1 & 4 & | & 2 \\ 0 & 1 & | & \frac{1}{a-4} \end{bmatrix},$$

which has a unique solution. On the other hand, when $a = 4$ the system becomes

$$\begin{bmatrix} 1 & 4 & | & 2 \\ 0 & 0 & | & 8 \end{bmatrix}$$

which has no solution. Finally, when $a = -4$ the system becomes

$$\begin{bmatrix} 1 & 4 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}$$

which has infinitely many solutions.

3. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Use Gaussian-type elimination to find A^{-1}. In this computation, indicate the first point at which A is in row-echelon form and reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 1 & 2 & 3 & | & 0 & 1 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & -1 & 1 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & -1 & 1 & 0 \\ 0 & 0 & -1 & | & 1 & -1 & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 0 & | & 2 & -1 & 1 \\ 0 & 1 & 0 & | & 1 & -1 & 2 \\ 0 & 0 & 1 & | & -1 & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & -1 \\ 0 & 1 & 0 & | & 1 & -1 & 2 \\ 0 & 0 & 1 & | & -1 & 1 & -1 \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$. In this case, row-echelon form is first achieved in the second to last matrix, and reduced row-echelon form is first achieved in the last matrix.

(b) Use A^{-1} to find the solution to the following linear system:

$$x + y + z = 1$$

$$x + 2y + 3z = 1$$

$$y + z = 1$$

First write the system as $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Now because A is invertible, we may multiply both sides of this equation (on the left) by A^{-1}, to find that

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix},$$

so $x = 0, y = 2, z = -1$ solves the system.

4. Let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be the matrix transformation associated to the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 1 \\ 3 & 5 & -1 \end{bmatrix}.$$

(a) Describe the vectors in the range of f (i.e., what relationships, if any, must hold among the components of a vector in range(f)?).

Because f is a matrix transformation, we know that f is defined by saying that $f(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is any vector in $\mathbf{R}^3$. For a vector $\mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to be in the range of f, there must be a solution to the equation $A\mathbf{x} = \mathbf{b}$, which has augmented matrix

$$\begin{bmatrix} 1 & 3 & -1 & | & a \\ 0 & -2 & 1 & | & b \\ 3 & 5 & -1 & | & c \end{bmatrix}.$$

This matrix reduces as follows:

$$\begin{bmatrix} 1 & 3 & -1 & | & a \\ 0 & -2 & 1 & | & b \\ 3 & 5 & -1 & | & c \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & -1 & | & a \\ 0 & -2 & 1 & | & b \\ 0 & -4 & 2 & | & c-3a \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & -1 & | & a \\ 0 & -2 & 1 & | & b \\ 0 & 0 & 0 & | & c-3a-2b \end{bmatrix}.$$

This has a solution only if $c - 3a - 2b = 0$, or in other words if $c = 3a + 2b$. Thus the range of f consists of all vectors in $\mathbf{R}^3$ of the form $\begin{bmatrix} a \\ b \\ 3a+2b \end{bmatrix}$.

(b) Describe the vectors in the kernel of f (i.e., what relationships, if any, must hold among the components of a vector in kernel(f)?).

Here we are looking for solutions to the equation $A\mathbf{x} = \mathbf{0}$. As in part (a), the associated augmented matrix reduces as follows:

$$\begin{bmatrix} 1 & 3 & -1 & | & 0 \\ 0 & -2 & 1 & | & 0 \\ 3 & 5 & -1 & | & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & -1 & | & 0 \\ 0 & -2 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

We let $z = r$, and then find that $y = z/2 = r/2$ and $x = z - 3y = r - 3r/2 = -r/2$. Thus the kernel of f consists of vectors in $\mathbf{R}^3$ of the form $\begin{bmatrix} -r/2 \\ r/2 \\ r \end{bmatrix}$.

5. For each of the following five statements, indicate clearly whether it is true or false. For TWO of the statements, also do the following (in the space below): If the statement is true, then explain why. If the statement is false, then provide a counterexample.

(a) ______ If $AB = O$, then either $A = O$ or $B = O$ (O is the zero matrix of the appropriate size).

FALSE. For example, let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then $AB = O$.

(b) ______ If $\mathbf{x}_p$ is a solution to the linear system $A\mathbf{x} = \mathbf{b}$, then so is $\mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is any solution to the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

TRUE. We are told that $A\mathbf{x}_p = \mathbf{b}$ and that $A\mathbf{x}_h = \mathbf{0}$, so then $A(\mathbf{x}_p + \mathbf{x}_h) = A\mathbf{x}_p + A\mathbf{x}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$, so $\mathbf{x}_p + \mathbf{x}_h$ is a solution to $A\mathbf{x} = \mathbf{b}$.

(c) ______ If A is a square matrix, then $A + A^T$ is symmetric.

TRUE. Note that $(A + A^T)^T = A^T + A^{T^T} = A^T + A = A + A^T$.

(d) ______ For square matrices A and B, $(A + B)^2 = A^2 + 2AB + B^2$.

FALSE. $(A + B)^2 = (A + B)(A + B) = A^2 + AB + BA + B^2$. To find a specific counterexample, one only needs to find two matrices A and B so that $AB \neq BA$, of which there are lots of choices.

(e) ______ Homogeneous linear systems are always consistent.

TRUE. The homogeneous system $A\mathbf{x} = \mathbf{0}$ always has the solution $\mathbf{x} = \mathbf{0}$.