Linear Algebra I **Test 2 Solutions** **February 25, 2005**

1. (a) Verify that the 2×1 matrices of the form $\begin{bmatrix} x \\ -x \end{bmatrix}$ form a subspace of $\mathbf{R}^2$.

Let $\mathbf{u} = \begin{bmatrix} a \\ -a \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} b \\ -b \end{bmatrix}$. Then

$$\mathbf{u} \oplus \mathbf{v} = \begin{bmatrix} a \\ -a \end{bmatrix} + \begin{bmatrix} b \\ -b \end{bmatrix} = \begin{bmatrix} a+b \\ -a-b \end{bmatrix} = \begin{bmatrix} (a+b) \\ -(a+b) \end{bmatrix}.$$

Matrices of this type are therefore closed under addition. Also for any real number c we have

$$c \odot \mathbf{u} = c \begin{bmatrix} a \\ -a \end{bmatrix} = \begin{bmatrix} ca \\ -ca \end{bmatrix},$$

so matrices of this type are closed under scalar multiplication. It follows that these matrices form a subspace of $\mathbf{R}^2$.

(b) Are the matrices of the form $\begin{bmatrix} x \\ x^2 \end{bmatrix}$ a subspace of $\mathbf{R}^2$? Justify your answer.

No. For example, $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, which is not the right form. Thus matrices of this form are not closed under addition, and so they cannot form a subspace.

2. Suppose V is the set of all 2×1 matrices $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x > 0$. It is easy to show that V is not a vector space using the usual operations of $\mathbf{R}^2$, but it *is* a vector space with the following operations:

$$\text{if } \mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix} \ (a > 0), \quad \mathbf{v} = \begin{bmatrix} s \\ t \end{bmatrix} \ (s > 0), \text{ and } c \text{ is a real number,}$$

then we define

$$\mathbf{u} \oplus \mathbf{v} = \begin{bmatrix} as \\ b+t \end{bmatrix}, \text{and } c \odot \mathbf{u} = \begin{bmatrix} a^c \\ cb \end{bmatrix}.$$

(a) With these operations, what is $\mathbf{0}$ in V? Explain. (Recall that $\mathbf{0}$ is defined to be the vector in V with the property that $\mathbf{u} \oplus \mathbf{0} = \mathbf{u}$ for all vectors $\mathbf{u}$ in V.)

The zero vector is some vector in W, say $\mathbf{0} = \begin{bmatrix} x \\ y \end{bmatrix}$ for some x and y. We need to find what x and y are. By definition of $\mathbf{0}$, we have that $\mathbf{u} \oplus \mathbf{0} = \mathbf{u}$ for any $\mathbf{u}$, so we have

$$\begin{bmatrix} a \\ b \end{bmatrix} \oplus \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ b+y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

By equating entries, we see that $x = 1$ and $y = 0$, so that $\mathbf{0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) Property (5) of vector spaces states that for all real numbers c and all vectors $\mathbf{u}$ and $\mathbf{v}$, we have $c \odot (\mathbf{u} \oplus \mathbf{v}) = (c \odot \mathbf{u}) \oplus (c \odot \mathbf{v})$. Verify that this property holds for V with the operations defined above.

For the left hand side we have

$$c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot \begin{bmatrix} as \\ b+t \end{bmatrix} = \begin{bmatrix} (as)^c \\ c(b+t) \end{bmatrix}.$$

For the right hand side we have

$$(c \odot \mathbf{u}) \oplus (c \odot \mathbf{v}) = \begin{bmatrix} a^c \\ cb \end{bmatrix} \oplus \begin{bmatrix} s^c \\ ct \end{bmatrix} = \begin{bmatrix} a^c s^c \\ cb + ct \end{bmatrix}.$$

The left and right sides are equal, so the property is verified.

3. Showing your work, find a basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbf{R}^3$ that includes the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

We adjoin the standard basis to the given vectors to form the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -3 & 1 & 0 & 1 & 0 \\ 6 & -2 & 0 & 0 & 1 \end{bmatrix}.$$

We then row reduce this matrix obtaining, for instance,

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}.$$

From this it is clear that the reduced row echelon form will have initial ones in columns one, two and four. Thus vectors one, two and four form a basis for $\mathbf{R}^3$. The basis is thus

$$\begin{bmatrix} 1 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

4. Let W be the subspace of M_{22} consisting of all symmetric 2×2 matrices.

(a) Find a finite set of matrices in W which span W. (Note that a symmetric 2×2 matrix must be of the form $\begin{bmatrix} x & y \\ y & z \end{bmatrix}$ for some real numbers x, y, z.)

Every matrix in W can be written in the form

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} = x \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It follows that the set of matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

spans W.

(b) Verify that the following set of vectors in W is linearly independent:

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

They are linearly independent precisely if the equation

$$a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{0}$$

has only the trivial solution $a = b = c = 0$. This vector equation is equivalent to the equation

$$a \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which in turn is equivalent to the system of equations

$$\begin{array}{ccccccc} a & & & + & c & = & 0 \\ a & + & b & & & = & 0 \\ a & + & b & & & = & 0 \\ & & & & c & = & 0 \end{array}$$

The fourth equation says that $c = 0$, from which equation one says $a = 0$, from which equations two and three say $b = 0$. We deduce that the vectors are linearly independent.

(c) What dimension is W? Explain (based on parts (a) and (b) above).

From part (a), there is a set of three vectors which spans W. From this we deduce that the dimension of W is at most 3. From part (b), there is a set of three vectors in$W$ which is linearly independent, from which we deduce that the dimension of W is at least three. Thus the dimension of W is three.

5. For each of the following eight statements, indicate clearly whether it is true or false. **For all problems, assume that V is a *finite-dimensional* vector space.**

(a) ______ For any vector $\mathbf{u}$ in V, it is always true that $0 \odot \mathbf{u} = \mathbf{0}$.

True. This was a theorem from shortly after vector spaces were defined. It follows directly from the properties vector spaces are required to have.

(b) ______ A set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of linearly independent vectors in V is necessarily a basis for the subspace they span.

True. Let W be the subspace they span. Then the vectors span W, and they are linearly independent. Thus they are a basis for the subspace W.

(c) ______ If S_1 is a linearly independent set of vectors in V, then any set S_2 containing S_1 is also linearly independent.

False. In particular, if S_2 has more vectors than the dimension of V, then is surely cannot be linearly independent.

(d) ______ If $\dim(V) = n$, then any set of n or more vectors must span V.

False. They could all be multiples of a single vector, for instance (where $n > 1$).

(e) ______ Any two bases for V must contain the same number of vectors.

True. This is a theorem, and it's the reason that it makes sense to talk about the dimension of V.

(f) ______ Every set that spans V contains a basis for that vector space.

True. This is a theorem, and we know how to find the basis in the case that V is $\mathbf{R}^n$.

(g) ______ Suppose W is a subspace of V and that S is a basis for V. Then some subset of S is a basis for W.

False. For example, the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form a basis for $\mathbf{R}^2$ (think of length one vectors pointing in the positive direction along the x and y axes). The vectors of the form $\begin{bmatrix} x \\ x \end{bmatrix}$ (i.e., the line $y = x$) is a subspace of $\mathbf{R}^2$ that doesn't even contain either of the basis vectors, so they certainly cannot form a basis for the subspace.

(h) ______ The dimension of the null-space of a matrix A (i.e., the kernel of the associated matrix transformation) is the number of arbitrary constants in the solution to the linear system $A\mathbf{x} = \mathbf{0}$.

True. We said in class that the the spanning set obtained by pulling out the constants in the general solution will always form a basis in this type of example. Thus the dimension of the kernel is the number of vectors in a basis, which is the number of vectors in this spanning set, which is the number of arbitrary constants in the general solution.