Linear Algebra I **Test 3 Solutions** **March 28, 2005**

1. Use the Gram-Schmidt process on $\mathbf{R}^3$ (where the inner product is the usual dot product) to orthonormalize the following basis (remember to clear fractions as you go and then normalize at the end):

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}$$

Denote the three vectors above as $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. We obtain an orthogonal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as follows:

$$\mathbf{v}_1 = \mathbf{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 = \begin{bmatrix}1\\0\\2\end{bmatrix} - \frac{3}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\mathbf{v}_2 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 = \begin{bmatrix}0\\2\\0\end{bmatrix} - \frac{-2}{2}\begin{bmatrix}0\\-1\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-2/3\\1/3\\1/3\end{bmatrix} \rightsquigarrow \begin{bmatrix}-2\\1\\1\end{bmatrix}$$

We now normalize these vectors (i.e., divide them by their lengths) to obtain the orthonormal basis

$$\left\{ \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix} \right\}$$

2. Find the Fourier polynomial of degree two for the function $f(t) = \cos^2(t)$ using the following facts:

$$\int_{-\pi}^{\pi} \cos^3(t)\,dt = 0 \qquad \int_{-\pi}^{\pi} \cos^2(t)\,\sin(t)\,dt = 0$$

$$\int_{-\pi}^{\pi} \cos^2(t)\,\cos(2t)\,dt = \pi/2 \qquad \int_{-\pi}^{\pi} \cos^2(t)\,\sin(2t)\,dt = 0$$

We find the Fourier polynomial of degree two by projecting $\cos^2(t)$ onto the subspace with orthogonal basis $\{1, \cos(t), \sin(t), \cos(2t), \sin(2t)\}$. Recall that the inner product is given by

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt.$$

We thus obtain

$$\cos^2(t) \approx \frac{\langle \cos^2(t), 1\rangle}{\langle 1, 1\rangle} + \frac{\langle \cos^2(t), \cos(t)\rangle}{\langle \cos(t), \cos(t)\rangle}\cos(t) + \frac{\langle \cos^2(t), \sin(t)\rangle}{\langle \sin(t), \sin(t)\rangle}\sin(t)$$
$$+ \frac{\langle \cos^2(t), \cos(2t)\rangle}{\langle \cos(2t), \cos(2t)\rangle}\cos(2t) + \frac{\langle \cos^2(t), \sin(2t)\rangle}{\langle \sin(2t), \sin(2t)\rangle}\sin(2t)$$

$$= \frac{\int_{-\pi}^{\pi} \cos^2(t)\,dt}{2\pi} + \frac{\int_{-\pi}^{\pi} \cos^2(t)\cos(t)\,dt}{\pi}\cos(t) + \frac{\int_{-\pi}^{\pi} \cos^2(t)\sin(t)\,dt}{\pi}\sin(t)$$
$$+\frac{\int_{-\pi}^{\pi} \cos^2(t)\cos(2t)\,dt}{\pi}\cos(2t) + \frac{\int_{-\pi}^{\pi} \cos^2(t)\sin(2t)\,dt}{\pi}\sin(2t).$$

Using the integral values given above, this becomes

$$\frac{\pi}{2\pi} + \frac{\pi/2}{\pi}\cos(2t) = \frac{1}{2} + \frac{1}{2}\cos(2t).$$

Note that this approximation happens to be equality in this case, as this is just a double-angle formula (i.e., all higher degree Fourier approximations are equal to this one).

3. Consider the inner product on $\mathbf{R}^2$ defined as follows:

$$\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \rangle = ac - 2ad - 2bc + 5bd.$$

(a) Find the angle between $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ using this inner product.

First note that

$$\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle = 1 - 0 - 0 + 0 = 1$$
$$\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rangle = 1 - 2 - 2 + 5 = 2$$
$$\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rangle = 1 - 2 - 0 + 0 = -1.$$

For the angle formula we now have

$$\cos(\theta) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\| \mathbf{u} \| \| \mathbf{v} \|} = \frac{-1}{\sqrt{1}\sqrt{2}} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2},$$

which implies that

$$\theta = \frac{3\pi}{4}.$$

(b) State the Cauchy-Schwarz inequality for an arbitrary vector space V with inner product $\langle\ ,\ \rangle$.

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \| \mathbf{u} \| \| \mathbf{v} \|$$

4. Suppose $L : V \to W$ is a linear transformation, where the dimension of V is n and the dimension of W is m.

(a) Suppose $n = 3$. If L is one-to-one, what is the dimension of the range of L?

The rank-nullity formula states that

$$\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(V).$$

We are told that $\dim(V) = 3$, and because L is one-to-one, we know that

$$\dim(\ker(L)) = 0,$$

so we now have

$$0 + \dim(\text{range}(L)) = 3,$$

or

$$\dim(\text{range}(L)) = 3.$$

(b) Suppose L is both one-to-one and onto. What can you say about the relationship between n and m? Explain.

Agin, the rank-nullity theorem states that

$$\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(V).$$

The fact that L is one-to-one tells us that

$$\dim(\ker(L)) = 0,$$

and the fact that L is onto tells us that

$$\dim(\text{range}(L)) = \dim(W) = m.$$

So we now have

$$0 + m = n,$$

or

$$m = n.$$

(c) Suppose $n = m$. Show that L is one-to-one if and only if L is onto.

Suppose $n = m$ and L is one-to-one. Then $\dim(\ker(L)) = 0$, so the rank-nullity theorem now says

$$\dim(\text{range}(L)) = \dim(V).$$

But

$$\dim(V) = n = m = \dim(W)$$

so we have

$$\dim(\text{range}(L)) = \dim(W),$$

which means that L is onto.

Now suppose $n = m$ and L is onto. Then $\dim(\text{range}(L)) = \dim(W)$, so the rank-nullity theorem now says

$$\dim(\ker(L)) + \dim(W) = \dim(V).$$

But because $\dim(W) = \dim(V)$, we deduce that

$$\dim(\ker(L)) = 0,$$

which means that L is one-to-one.

5. Verify the rank-nullity theorem for the linear transformation $L : M_{22} \to M_{22}$ defined by

$$L(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A - A \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

We first write out L more explicitly. For any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$\begin{aligned} L(A) = L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} - \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = \begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix}. \end{aligned}$$

To describe the kernel of L, we set this output equal to the zero matrix and see what relationships must hold among the original entries of A. Thus we have

$$\begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \Rightarrow \qquad \begin{cases} c = 0 \\ a = d \end{cases}$$

Thus matrices A that are in the kernel must have the form

$$A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}.$$

So we have

$$\ker(L) = \left\{\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\right\} = \left\{a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}.$$

These matrices are linearly independent, so the dimension of the kernel is two.

The range of L is all outputs of L, and so consists of all matrices of the form

$$L(A) = \begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix}.$$

We deduce that

$$\begin{aligned} \operatorname{range}(L) = \left\{\begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix}\right\} &= \left\{a\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\} \\ &= \operatorname{span}\left\{\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}. \end{aligned}$$

The first and last of these matrices are multiples of one another, so we discard one. The remaining two are linearly independent, and so form a basis for the range. Thus the range has dimension two.

The dimension of M_{22} is four, so the rank-nullity theorem is verified, as $2 + 2 = 4$.