Linear Algebra I Test 4 Review

Coordinates

Notation: In all that follows V is an n-dimensional vector space with two bases $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $S' = \{\mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$, and W is an m-dimensional vector space with two bases $T = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ and $T' = \{\mathbf{w}'_1, \ldots, \mathbf{w}'_m\}$. Finally, $L : V \to W$ is a linear map, and we let A be the matrix representation of L with respect to S and T, while B is the matrix representation of L with respect to S' and T'.

For all the examples, we have $V = P_2$ with bases

$$S = \{t^2, t, 1\} \text{ and } S' = \{t^2, t+1, t-1\},$$

and $W = M_{22}$ with bases

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and

$$T' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

We will use the linear map $L : P_2 \to M_{22}$ given by

$$L(at^2 + bt + c) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

Coordinate Vectors

Here is the definition of what a coordinate vector is:

$$\boxed{[\mathbf{v}]_S = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ means } \mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3}$$

Example: For $V = P_2$ we have

$$[t^2 + 2t - 4]_S = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}, \text{ because } t^2 + 2t - 4 = 1(t^2) + 2(t) - 4(1),$$

and

$$[t^2 + 2t - 4]_{S'} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \text{ because } t^2 + 2t - 4 = 1(t^2) - 1(t+1) + 3(t-1).$$

These were found by equating corresponding coefficients and solving the resulting system of linear equations.

Exercise: For $\mathbf{v} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ in $W = M_{22}$, find $[\mathbf{v}]_T$ and $[\mathbf{v}]_{T'}$.

Here is the fact that allows you to answer virtually all questions about vectors in V by instead answering the corresponing question about the corresponding coordinate vectors in $\mathbf{R}^n$:

$$L : V \to \mathbf{R}^n \text{ where } L(\mathbf{v}) = [\mathbf{v}]_S \text{ is an isomorphism}$$

Transition Matrices

Here is the definition of the transition matrix from S' to S:

$$\boxed{P_{S \leftarrow S'} = \begin{bmatrix} | & | & & | \\ [\mathbf{v}_1']_S & [\mathbf{v}_2']_S & \cdots & [\mathbf{v}_n']_S \\ | & | & & | \end{bmatrix}}$$

Note that $P_{S \leftarrow S'}^{-1} = P_{S' \leftarrow S}$.

Example: In $V = P_2$ we have that

$$[t^2]_{S'} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [t]_{S'} = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix} \qquad [1]_{S'} = \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

Thus the transition matrix from S to S' is

$$P_{S' \leftarrow S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix}.$$

Exercise: Find the transition matrices from S' to S, from T to T', and from T' to T directly.

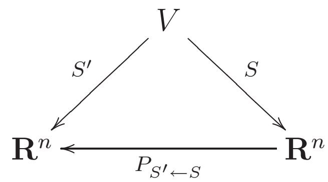
Here is what transition matrices are designed to do:

$$\boxed{[\mathbf{v}]_{S'} = P_{S' \leftarrow S}[\mathbf{v}]_S}$$

Example:

$$[t^2 + 2t - 4]_{S'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$

Here's the picture:



Matrix Representations

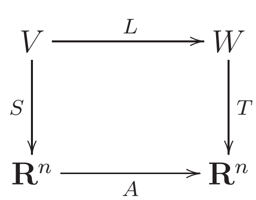
Here is the definition of the matrix representation of L with respect to S and T:

$$\boxed{A = \begin{bmatrix} | & | & & | \\ [L(\mathbf{v}_1)]_T & [L(\mathbf{v}_2)]_T & \cdots & [L(\mathbf{v}_n)]_T \\ | & | & & | \end{bmatrix}}$$

Special Case $V = W$: "The matrix representation of L with respect to S" means "the matrix representation of L with respect to S and S."

Here is what matrix representations are designed to do, with the corresponding picture:

$$\boxed{A[\mathbf{v}]_S = [L(\mathbf{v})]_T}$$



Example: The matrix representation of L with respect to S' and T' is found as follows. Applying L to each S'-vector, we have

$$L(t^2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad L(t+1) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad L(t-1) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

Now finding the T'-coordinates of these matrices we have (check this)

$$[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}]_{T'} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \qquad [\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}]_{T'} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \end{bmatrix}, \qquad [\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}]_{T'} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Thus the matrix representation of L with respect to S' and T' is the matrix

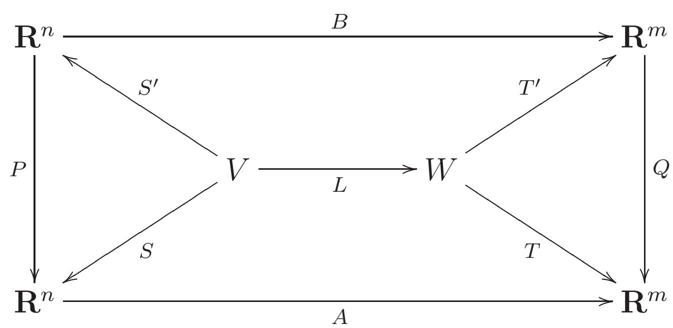
$$B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Exercise: Find the matrix representation of L with respect to S and T.

Changing Matrix Representations

Here is the relationship between A and B, along with the picture:

$$\boxed{B = Q^{-1}_{T \leftarrow T'} A P_{S \leftarrow S'}}$$



Example: The transition matrices from S' to S and from T to T' are (check these)

$$P_{S \leftarrow S'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \qquad Q^{-1}_{T \leftarrow T'} = Q_{T' \leftarrow T} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix},$$

while the matrix representation of L with respect to S and T is (check this)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exercise: Verify that the product $Q_{T' \leftarrow T} A P_{S \leftarrow S'}$ really is the matrix B we found above.

Special Case $V = W$: Given $L : V \to V$, if A is the matrix representation of L with respect to S and B is the matrix representation of L with respect to S', then

$$B = P^{-1}_{S \leftarrow S'} A P_{S \leftarrow S'}$$

We say two $n \times n$ matrices A and B are similar if there is some P so that

$$B = P^{-1}AP.$$

Theorem. *A and B are similar if and only if they represent the same linear operator L with respect to two bases S and S'.*

Properties of Determinants

You need not fully know nor fully understand the definition of the determinant found in Section 6.1, except that you should understand that the determinant is a sum (with + and − signs strewn about), and each summand has exactly one factor from each row and column. From this (along with understanding a bit about where the signs come from) we deduce all of the following (A and B are assumed to be square matrices of the same size):

1. $\det(A^T) = \det(A)$;
2. Switching two rows of A changes the sign of the determinant;
3. If A has two identical rows, then $\det(A) = 0$;
4. Multiplying a row of A by k multiplies the determinant by k;
5. If A has a row of zeros, then $\det(A) = 0$;
6. If a multiple of one row of A is added to another, the determinant is unchanged;
7. The determinant of an upper triangular matrix equals the product of the diagonal entries;
8. A is non-singular/invertible if and only if $\det(A) \neq 0$;
9. $A\mathbf{v} = \mathbf{0}$ has a non-trivial solution if and only if $\det(A) = 0$;
10. $\det(A)\det(B) = \det(AB)$;
11. If A is non-singular/invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$;
12. If A and B are similar, then $\det(A) = \det(B)$.

Method I: Definition (Leibniz' formula). This is not really practical, although it does easily give us the 2×2 case:

$$\boxed{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.}$$

Method II: Row reduction. This is ultimately the most practical method for arbitrary matrices, particularly those larger than 4×4. Given a matrix A, let D denote the determinant of A. We want to know D. We row reduce A until it is in upper triangular form, keeping track of how D is affected (according to properties 2, 4, 6 above). We then use property 7 above to find the determinant of the upper triangular matrix, and solve for D.

Example:

$$\begin{bmatrix} -2 & 2 & -3 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2 & 2 & -3 \\ 2 & -2 & -6 \\ 0 & 2 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2 & 2 & -3 \\ 0 & 0 & -9 \\ 0 & 2 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2 & 2 & -3 \\ 0 & 2 & -4 \\ 0 & 0 & -9 \end{bmatrix}$$

where the row reductions used are

- multiply row 2 by -2;
- add row 1 to row 2;
- interchange rows 2 and 3.

If we let D be the determinant of A, then the first step multiplies the determinant by -2, the second does nothing, and the third changes its sign. Thus the determinant of the final upper triangular matrix is $2D$. So we have $2D = (-2)(2)(-9) = 36$, so $D = 18$.

Method III: Cofactor expansion (Laplace's formula). Let M_{ij} be the $(n-1)\times(n-1)$ matrix obtained by removing the ith row and jth column from A. We define the ijth cofactor of A to be the quantity

$$A_{ij} = (-1)^{i+j}\det(M_{ij}).$$

We then have the ith row cofactor expansion:

$$\det(A) = \sum_{j=1}^{n} a_{ij}A_{ij}.$$

This reduces an $n \times n$ determinant to an alternating sum of $(n-1) \times (n-1)$ determinants. Repeating this reduction, we can get any determinant down to a bunch of 2×2 determinants and use the formula $ad - bc$ obtained from Method I. This is generally only useful for 3×3 and occasionally for 4×4 matrices.

Example: Expanding along the first row, we have

$$\det \begin{bmatrix} -2 & 2 & -3 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix} = -2 \left(\det \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \right) - 2 \left(\det \begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix} \right) + (-3) \left(\det \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} \right)$$

$$= -2(-1-0) - 2(1-6) - 3(0-2) = 18.$$

Method IV: ONLY WORKS FOR 3×3. Write the first two columns again to the right of the third column. Multiply down all the diagonals. The ones that go down left to right get positive signs, the ones that go down right to left get negative signs. Add the results. If you already know how to do it this way, feel free; otherwise, it's probably not worth the trouble.

Finding Inverses

Given a square matrix A, we know that A^{-1} exists exactly if $\det(A) \neq 0$. In fact, there is a nifty formula for A^{-1} that uses the determinant and the cofactors A_{ij} defined above. This formula is not very practical beyond 3×3 matrices, however. *NOTE THE "TRANSPOSED" POSITION OF THE COFACTORS!*

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}.$$

In particular, we have the following nice formula in the 2×2 case:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Exercise: Write down a bunch of matrices, ranging in size from 2×2 to at least 4×4, and find their determinants. For those with non-zero determinant, find the inverses.

Test Information

You should assume that there will be a multi-part problem much like the series of examples in the first portion of this review. In particular, you should know how to find the coordinates of a vector with respect to a given basis, you should know how to find the transition matrix from one basis to another, and you should know how to verify that the transition matrix does what it is supposed to do. You should also be able to find the matrix representation of a linear map with respect to given bases, and how to use transition matrices to find the matrix transformation of the same linear map with respect to different bases. You should know the basic properties of the determinant and know some way of finding a determinant of any size. *You will not be required to find a determinant in any prespecified way.* Finally, you should know how to find inverses using the cofactor method. There will be some true/false questions, mostly (but not entirely) on properties of determinants. You may have to verify/prove a fact or two.