Linear Algebra I **Test 4** **April 18, 2005**

PART I

Suppose $T = \{t^2,\ t,\ 1\}$ is the standard basis for the vector space P_2, and suppose $T' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is some other basis for P_2.

Ia. Find T', given that the transition matrix from T' to T is

$$Q_{T\leftarrow T'} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The columns of Q are exactly the T-coefficients of the T' vectors. Thus we have

$$\mathbf{v}_1 = 1t^2 + 1t + 1 = t^2 + t + 1 \qquad \mathbf{v}_2 = 0t^2 + 1t + 1 = t + 1 \qquad \mathbf{v}_3 = 0t^2 + 0t + 1 = 1,$$

so the basis is

$$T' = \{t^2 + t + 1,\ t + 1,\ 1\}.$$

Ib. Use cofactors to find the inverse of $Q_{T\leftarrow T'}$.

The cofactors are as follows:

$$Q_{11} = \det\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 1 \qquad Q_{21} = -\det\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 0 \qquad Q_{31} = \det\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0$$

$$Q_{12} = -\det\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = -1 \qquad Q_{22} = \det\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 1 \qquad Q_{32} = -\det\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 0$$

$$Q_{13} = \det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0 \qquad Q_{23} = -\det\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = -1 \qquad Q_{33} = \det\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 1$$

The determinant of Q is 1 (this is most easily seen by multiplying the diagonal elements). Thus we have

$$Q_{T\leftarrow T'}^{-1} = Q_{T'\leftarrow T} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Ic. Consider the linear map $L : M_{22} \to P_2$ defined as follows:

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b)t^2 + (a-c)t + (2d).$$

Find the matrix representation A of L with respect to the bases S and T directly, where S is the standard basis for M_{22}:

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

The columns of A are as follows:

$$\left[L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \right]_T = [t^2 + t]_T = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[L\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right)\right]_T = [t^2]_T = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\left[L\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right)\right]_T = [-t]_T = \begin{bmatrix}0\\-1\\0\end{bmatrix}$$

$$\left[L\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)\right]_T = [2]_T = \begin{bmatrix}0\\0\\2\end{bmatrix}$$

Thus the matrix representation is

$$A = \begin{bmatrix}1 & 1 & 0 & 0\\1 & 0 & -1 & 0\\0 & 0 & 0 & 2\end{bmatrix}$$

Suppose S' is the following basis for M_{22}:

$$S' = \left\{\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}.$$

Id. Find the transition matrix from S' to S.

The columns of the transition matrix are as follows:

$$\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}_S = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}_S = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}_S = \begin{bmatrix}1\\0\\0\\1\end{bmatrix} \qquad \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}_S = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Thus the transition matrix is

$$P_{S\leftarrow S'} = \begin{bmatrix}1 & 1 & 1 & 0\\1 & 1 & 0 & 0\\1 & 0 & 0 & 0\\1 & 0 & 1 & 1\end{bmatrix}$$

Ie. Use your answers from problems **I**a–d to find the matrix representation B of L with respect to S' and T' (i.e., find B indirectly using transition matrices and your matrix A from **I**c).

The formula is

$$B = Q_{T'\leftarrow T}AP_{S\leftarrow S'} = \begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0 & 0\\1 & 0 & -1 & 0\\0 & 0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 0\\1 & 1 & 0 & 0\\1 & 0 & 0 & 0\\1 & 0 & 1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}2 & 2 & 1 & 0\\-2 & -1 & 0 & 0\\2 & -1 & 1 & 2\end{bmatrix}$$

If. Find the S'-coordinates of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. (Hint: You might be able to get this by looking, rather than having to solve a system of equations.)

Note that

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so we have

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}_{S'} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Ig. Find $L\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$ two different ways: (i) using the matrix representation A with respect to the bases S and T; (ii) using the matrix representation B with respect to the bases S' and T'.

(i) Using A we have that

$$\left[L\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\right]_T = A\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}_S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix},$$

so we have

$$L\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right) = 3t^2 + t^4.$$

(ii) Using B we have that

$$\left[L\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\right]_{T'} = B\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}_{S'} = \begin{bmatrix} 2 & 2 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ 2 & -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix},$$

so we have

$$L\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right) = 3(t^2 + t + 1) - 2(t + 1) + 3 = 3t^2 + t + 4.$$

PART II

IIa. Find the determinants of the following matrices (look for shortcuts!).

$$\begin{bmatrix} -2 & 4 \\ -3 & 7 \end{bmatrix} \qquad \begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 1 & 5 & 3 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 5 & 3 & 2 \\ -3 & 1 & 8 & -3 & 1 \\ 0 & 9 & -4 & 2 & 9 \\ 1 & -1 & 0 & -1 & -1 \\ 6 & 4 & -3 & -2 & 4 \end{bmatrix}$$

(i) $(-2)(7) - (-3)(4) = -14 + 12 = -2$

(ii) $3\det\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix} - (-1)\det\begin{bmatrix}4 & 2\\3 & 2\end{bmatrix} + 2\det\begin{bmatrix}4 & 1\\3 & 1\end{bmatrix} = 3(2-2)+(8-6)+2(4-3) = 4$

(iii) $(4)(2)(-3)(5) = -120$

(iv) The determinant is 0 because columns two and five are equal.

IIb. Find all values of λ for which the matrix equation $B\mathbf{v} = \mathbf{0}$ has a nontrivial solution, where

$$B = \begin{bmatrix}1-\lambda & 2\\ 1 & -\lambda\end{bmatrix}.$$

The equation has a nontrivial solution precisely when $\det B = 0$. So we have

$$\det B = (1-\lambda)(-\lambda) - (2)(1) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0,$$

which has solutions

$$\lambda = 2,\ -1.$$

IIc. True/False. For all questions, assume that A and B are both 3×3 matrices.

(i) ______ $\det(A+B) = \det(A) + \det(B)$

False; consider $A = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and $B = -A$, for instance.

(ii) ______ $\det(B^{-1}A) = \dfrac{\det(A)}{\det(B)}$

True; $\det(B^{-1}A) = \det(B^{-1})\det(A) = \dfrac{\det(A)}{\det(B)}$.

(iii) ______ If $\det(A) = 0$ then A has at least two equal rows.

False; there are matrices with determinant zero and with no equal rows (in fact, there's one in the determinant problem above).

(iv) ______ If A has a column of zeros, then $\det(A) = 0$.

True; a row of zeros implies zero determinant, and the determinant of A is the same as that of A^T.

(v) ______ A is singular if and only if $\det(A) = 0$.

True; this is one of the most important properties of the determinant.

(vi) ______ If B is the reduced row echelon form of A, then $\det(B) = \det(A)$.

False; any matrix with nonzero determinant row reduces to the identity, which has determinant one.