Week 14 homework

17.8 #3: The boundary curve C is the circle $x^2+y^2=4$, $z=0$, oriented in the counterclockwise direction. A parametrization is $\mathbf{r}(t)=\langle 2\cos t, 2\sin t, 0\rangle$, $0\le t\le 2\pi$, so $\mathbf{r}'(t)=\langle -2\sin t, 2\cos t, 0\rangle$ and $\mathbf{F}(\mathbf{r}(t))=\langle 4\cos^2 t, 4\sin^2 t, 0\rangle$. Applying Stokes' theorem, we have

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt=\int_0^{2\pi}(-8\cos^2 t\sin t+8\sin^2 t\cos t)\,dt=0.$$

17.8 #7: Note that $\operatorname{curl}\mathbf{F}=\langle -2z,-2x,-2y\rangle$, and we take the surface S to be the planar region enclosed by C, so S is a portion of the plane $x+y+z=1$ parametrized via $\mathbf{r}(u,v)=\langle u,v,1-u-v\rangle$ over the triangular domain D, where $0\le u\le 1$ and $0\le v\le 1-u$. Note that $\mathbf{r}_u\times\mathbf{r}_v=\langle 1,1,1\rangle$ (since C is oriented counterclockwise, we orient S upward). Now we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r}=\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}&=\iint_D \langle 2u+2v-2,-2u,-2v\rangle\cdot\langle 1,1,1\rangle\,dA\\&=\int_0^1\int_0^{1-u}-2\,dv\,du=-1.\end{aligned}$$

17.8 #14: The plane intersects the coordinate axes at $x=1$, $y=z=2$, so the boundary curve C consists of the three line segments $\mathbf{r}_1(t)=\langle 1-t,2t,0\rangle$, $\mathbf{r}_2(t)=\langle 0,2-2t,2t\rangle$, and $\mathbf{r}_3(t)=\langle t,0,2-2t\rangle$. Thus

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r}&=\int_0^1\langle 1-t,2t,0\rangle\cdot\langle -1,2,0\rangle\,dt+\int_0^1\langle 0,2-2t,0\rangle\cdot\langle 0,-2,2\rangle\,dt\\&\quad+\int_0^1\langle t,0,0\rangle\cdot\langle 1,0,-2\rangle\,dt=0.\end{aligned}$$

Now $\operatorname{curl}\mathbf{F}=\langle xz,-yz,0\rangle$, while with the usual parametrization $\mathbf{r}(u,v)=\langle u,v,2-2u-v\rangle$ (where D is where $0\le u\le 1$ and $0\le v\le 2-2u$) we have that $\mathbf{r}_u\times\mathbf{r}_v=\langle 2,1,1\rangle$, so that

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}&=\iint_D\langle 2u-2u^2-uv,-2v+2uv+v^2,0\rangle\cdot\langle 2,1,1\rangle\,dA\\&=\int_0^1\int_0^{2-2u}(4u-4u^2-2v+v^2)\,dv\,du=0.\end{aligned}$$