# INTEGRALS OVER SURFACES $S$

An integral $\iint_S f\,dS$ over a surface $S$ simply adds up numbers over the surface, where the numbers are provided by the function $f$. To calculate such an integral, we need to have a parametrization $\mathbf{r}(u,v)$, over some domain $D$ in the $uv$-plane, of the surface $S$. We then compute using the following:

$$\iint_S f\,dS = \iint_D f(\mathbf{r}(u,v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA.$$

## Special Cases:

- When $S$ is some region $D$ of the $xy$-plane, we use parameters $u = x$ and $v = y$, and the expression becomes

  $$\iint_S f\,dS = \iint_D f(x,y)\,dA.$$

  One often takes the function $f$ to represent height (measured via the $z$-axis), so that this integral gives (signed) volume under the graph $z = f(x,y)$. This is the usual double integral from earlier this semester.

- When the function $f$ is identically 1, this integral simply adds up the $\Delta S$ bits, and hence gives the *surface area* of $S$. Thus we have

  $$\text{Area}(S) = \iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA.$$

- When $S$ is the graph of a function of the form $z = f(x,y)$, note that we have

  $$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

- In the presence of a vector field $\mathbf{F}$ in space, we often consider the function $f$ defined as the normal component of $\mathbf{F}$ across $S$; i.e., $f = \mathbf{F}\cdot\mathbf{n}$, where $\mathbf{n}$ is the *unit* normal vector $\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}$ to the surface $S$, so that the integral of this function across $S$ becomes

  $$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(\mathbf{F}(\mathbf{r}(u,v,))\cdot\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\right)|\mathbf{r}_u \times \mathbf{r}_v|\,dA = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA,$$

  where the left-most expression is just shorthand notation for the case where this particular function is the one being integrated. This is such a useful thing to do that we simply call it "integrating the vector field across the surface," and the quantity thus obtained is called the *flux* of the vector field across the surface.

> *Note:* For this to be defined, the surface needs to be orientable, so that there is a consistent choice of normal vector onto which to project. Roughly speaking, you need to be able to color one side of the surface red and the other side blue, without ever accidentally coloring anything purple.