Quiz 10 (Thursday, Apr 27)

Use Stokes' theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = \langle -y^2, x, z^2 \rangle$ and S is the three non-horizontal sides of the tetrahedron with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Solution: We use Stokes' theorem to replace the surface S with the horizontal base S' of the tetrahedron, which is the triangle with corners $(0, 0, 0)$, $(1, 0, 0)$, and $(0, 1, 0)$. The curl of $\mathbf{F}$ is $\langle 0, 0, 1 + 2y \rangle$, and the upward normal for the new surface is $\langle 0, 0, 1 \rangle$. We thus have

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S'} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_0^{1-x} \langle 0, 0, 1 + 2y \rangle \cdot \langle 0, 0, 1 \rangle \, dy \, dx$$

$$= \int_0^1 \int_0^{1-y} (1 + 2y) \, dy, dx = \frac{5}{6}.$$