Calculus IV Test 1 Solutions February 14, 2006

1. Complete the definition: A function $f(x,y)$ is *continuous* at (a,b) if:

(a) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists
(b) $f(a,b)$ is defined, and
(c) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

2. The following is a contour map for the function $f(x,y)$.

(a) Estimate the value of f at the point $(1, \frac{1}{3})$.

As $(1, 1/3)$ lies right on the contour curve $f = 4$, the value is 4.

(b) Is $f_x(1, \frac{1}{3})$ positive, negative, or approximately zero?

It is positive, because the next contour line in the positive x direction corresponds to a larger function value.

(c) What about $f_y(1, \frac{1}{3})$?

It is negative, because the next contour line in the positive y direction corresponds to a smaller function value.

(d) What about $D_{\mathbf{u}}(1, \frac{1}{3})$, where $\mathbf{u} = \langle 1, 1 \rangle$?

It is approximately zero, because this direction (north-east) is roughly tangent to the contour curve, and thus represents a direction of unchanging function values.

(e) Which is largest (in absolute value)?

It appears that $f_x(1, \frac{1}{3})$ is the largest, because its next contour curve is closer than that of f_y.

(f) At all marked points, sketch the gradient vector for the function, keeping in mind their relative lengths.

Your arrows should be perpendicular to the contour curves, pointing in the direction of increasing function values. Moreover, the nearer the next contour curve in that direction, the longer your arrow should be.

3. State the extreme value theorem for functions of two variables.

If $f(x,y)$ is differentiable and defined on a closed and bounded domain D, then there are points in the domain where f attains its absolute maximum and absolute minimum on that domain.

4. Use polar coordinates to show that $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$.

Substituting polar coordinates, we obtain

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)}{r} = \lim_{r\to 0} r\cos\theta\sin\theta.$$

Note that

$$-r \le r\cos\theta\sin\theta \le r,$$

for all values of θ and $r \ge 0$. Because the limit, as r approaches zero, of the outer two functions is zero, the squeeze theorem implies the limit of the inside function is zero.

5. Write an explicit system of equations that can be used to optimize $f(x,y) = x^2y$ subject to the constraint $x^2 + y^3 = 4$. Do not solve the system.

Letting $g(x,y) = x^2 + y^3$, we first calculate

$$\nabla f(x,y) = \langle 2xy, x^2\rangle, \qquad \nabla g = \langle 2x, 3y^2\rangle.$$

Thus the system of equations we need to solve is

$$2xy = 2\lambda x, \qquad x^2 = 3\lambda y^2, \qquad x^2 + y^3 = 4.$$

6. Suppose the temperature in the room is given by $T(x,y,z) = x^2y + y^2z - z^2x$.

(a) Find the equation for the plane tangent to the level surface $T(x,y,z) = 1$ at the point $(-1,0,1)$.

We first calculate

$$\nabla T(-1,0,1) = \langle 2xy - z^2, x^2 + 2yz, y^2 - 2xz\rangle|_{(-1,0,1)} = \langle -1, 1, 2\rangle.$$

Now because ∇T is perpendicular to level sets of T, this vector is normal to the tangent plane we seek. The plane is thus given by

$$-1(x+1) + 1(y-0) + 2(z-1) = 0.$$

(b) Find the direction in which one should travel from $(-1,0,1)$ in order to *decrease* temperature as fast as possible.

The direction of fastest decrease is the negative gradient, which is $\langle 1, -1, -2\rangle$.

(c) Find the directional derivative of T at $(-1,0,1)$ in the direction of $\mathbf{v} = \langle 1,1,1\rangle$.

Because $\langle 1,1,1\rangle$ has length $\sqrt{3}$, we use the unit vector $\mathbf{u} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle$, and compute

$$D_{\mathbf{u}}f = \nabla f(-1,0,1)\cdot \mathbf{u} = \langle -1,1,2\rangle \cdot \frac{1}{\sqrt{3}}\langle 1,1,1\rangle = \frac{2}{\sqrt{3}}.$$

7. Let $f(x,y) = x^2 + y^2 + kxy$.

(a) Show that $(0,0)$ is always a critical point for f, regardless of the value of k.

Note that $f_x(0,0) = 2x + ky|_{(0,0)} = 0$ and $f_y(0,0) = 2y + kx|_{(0,0)} = 0$.

(b) Classify this critical point for all possible values of k, being sure to indicate for which k the second derivative test fails.

We calculate $f_{xx} = 2$, $f_{yy} = 2$ and $f_{xy} = k$, so that $D = 4 - k^2$. Thus when $k = \pm 2$ we have $D = 0$ and the second derivative test fails. When $-2 < k < 2$, we have $D > 0$, and because $f_{xx} = 2 > 0$, the critical point is a minimum. When $k < -2$ or $k > 2$, we have $D < 0$, so the critical point is a saddle.

8. If x and y are the lengths of two sides of a triangle, and the angle they contain is θ, then the area of the triangle is given by

$$A = \frac{1}{2}xy\sin\theta.$$

Suppose you measure a triangle and find that $x = 24$ in., $y = 20$ in., and $\theta = \pi/6$ rad., calculating an area of 60 square inches. Now suppose your measurements of x and y were only accurate up to an error of 0.1 inches, while your measurement for θ was accurate up to an error of $\frac{1}{100\sqrt{3}}$ radians. Estimate the maximum error in the calculated area of the triangle at $t = 0$.

First note that

$$\frac{\partial A}{\partial x} = \frac{1}{2}y\sin\theta, \qquad \frac{\partial A}{\partial y} = \frac{1}{2}x\sin\theta, \qquad \frac{\partial A}{\partial \theta} = \frac{1}{2}xy\cos\theta.$$

Evaulating these at the measured values gives

$$\frac{\partial A}{\partial x} = 5, \qquad \frac{\partial A}{\partial y} = 6, \qquad \frac{\partial A}{\partial \theta} = 120\sqrt{3}.$$

We thus have

$$\Delta A \approx dA = \frac{\partial A}{\partial x}\Delta x + \frac{\partial A}{\partial y}\Delta y + \frac{\partial A}{\partial \theta}\Delta\theta = 5(0.1) + 6(0.1) + \frac{1}{100\sqrt{3}}(120\sqrt{3}) = 2.3.$$

(b) Now suppose your triangle is changing size with time, so that x, y, and θ are all functions of t. Your measurements reveal that when $t = 0$, x is increasing at a rate of 2 in/s, y is decreasing at a rate of 3 in/s, and θ is increasing at a rate of $\frac{1}{20}$ rad/s. Is the area of the triangle increasing or decreasing when $t = 0$? At what rate?

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt} = 5(2) + 6(-3) + 120\sqrt{3}(\frac{1}{20}) = 6\sqrt{3} - 8 > 0.$$

`Thus the area is increasing at the rate of` $6\sqrt{2}-8$ `square inches per second.`

9. Verify that Clairaut's theorem holds for $f(x, y) = \sin(x^2y)$.

$$f_x = 2xy\cos(x^2y) \quad \Rightarrow \quad f_{xy} = 2x\cos(x^2y) - 2x^3y\sin(x^2y).$$
$$f_y = x^2\cos(x^2y) \quad \Rightarrow \quad f_{yx} = 2x\cos(x^2y) - 2x^3y\sin(x^2y).$$