Calculus IV   Test 3 Solutions   April 20, 2006   Name:

1. A wire in space traces out the curve described by $\mathbf{r}(t) = \langle t^2, t^3, t^2 \rangle$, for $1 \le t \le 2\pi$. Find the total mass of the wire given that the density at the point (x, y, z) is given by

$$f(x,y,z) = \sqrt{8\left(\frac{xz}{y}\right)^2 + 9xz}.$$

We have that

$$|\mathbf{r}'(t)| = \sqrt{(2t)^2 + (3t^2)^2 + (2t)^2} = \sqrt{8t^2 + 9t^4}.$$

Also note that

$$f(\mathbf{r}(t)) = \sqrt{8\left(\frac{t^2 t^2}{t^3}\right)^2 + 9t^2 t^2} = \sqrt{8t^2 + 9t^4}.$$

Thus

$$M = \int_1^2 f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt = \int_1^2 \left(8t^2 + 9t^4\right) dt = \frac{56}{3} + \frac{279}{5}.$$

2. (a) State and prove the fundamental theorem for line integrals (you may do it either for two or three dimensions).

The fundamental theorem of line integrals states that

$$\int_a^b \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

To prove it, we note that

$$\int_a^b \nabla f \cdot d\mathbf{r} = \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right) dt$$

$$= \int_a^b \left(\frac{d}{dt} f(\mathbf{r}(t)) \right) dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

(b) Let C be the curve $\mathbf{r}(t) = \langle e^{t\sin(\pi t)}, t - t^5 \rangle$ for $0 \le t \le 1$. Verify that

$$\int_C (2e^{2x}\sin y - e^{x^2})\,dx + (e^{2x}\cos y + \ln(y^2+1))\,dy = 0,$$

by showing that this is the integral of a conservative vector field along a closed loop. *Do not find a potential function for the vector field.*

To see that the vector field is conservative, we note that

$$\frac{\partial Q}{\partial x} = 2e^{2x}\cos y = \frac{\partial P}{\partial y},$$

and that $\mathbf{F}$ is defined everywhere. Also, because

$$\mathbf{r}(1) = \langle 1, 0 \rangle = \mathbf{r}(0),$$

the curve is closed.

3. (a) Find a potential function for $\mathbf{F}(x, y) = \langle 2e^y \cos(2x) + 2xy, e^y \sin(2x) + x^2 \rangle$.

Given that $f_x = 2e^y \cos(2x) + 2xy$, we deduce that

$$f(x, y) = e^y \sin(2x) + x^2 y + g(y),$$

for some function g. But now because $f_y = e^y \sin(2x) + x^2$, we deduce that $g'(y) = 0$, so g is constant. We thus find a potential function of

$$f(x, y) = e^y \sin(2x) + x^2 y.$$

(b) Let C be the curve $\mathbf{r}(t) = \langle t \cos^2 t, e^{t^2 \sin t} \rangle$ for $0 \le t \le \pi$, and let $\mathbf{F}$ be the vector field from part (a) above. Use your answer from part (a) to find the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ without directly calculating the integral.

Applying the fundamental theorem of line integrals, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = f(\pi, 1) - f(0, 1) = \pi^2.$$

4. Let C be the portion of the graph of $y = x^2 + x - 2$ where $0 \le x \le 1$, and let $\mathbf{F}(x, y) = \langle y - x, x^2 \rangle$. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

The curve is not closed, so we cannot use the fundamental theorem. The vector field is not conservative, so we cannot use Green's theorem. We calculate directly, using the parametrization

$$\mathbf{r}(t) = \langle t, t^2 + t - 2 \rangle.$$

Thus we have

$$\mathbf{r}'(t) = \langle 1, 2t + 1 \rangle,$$

and

$$\mathbf{F}(\mathbf{r}(t)) = \langle t^2 - 2, t^2 \rangle.$$

So now

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle t^2 - 2, t^2 \rangle \cdot \langle 1, 2t + 1 \rangle \, dt = \int_0^1 (2t^2 + 2t^3 - 2) \, dt = -\frac{5}{6}.$$

5. (a) State the divergence form of Green's theorem.

$$\int_{\partial D} (\mathbf{F} \cdot \mathbf{n}) \, ds = \iint_D (\texttt{div}\, \mathbf{F}) \, dA.$$

(b) Let R be the region in the upper half-plane $y \geq 0$ inside the circle $x^2 + y^2 = 4$. Let C be the boundary of R, oriented counterclockwise. Let $\mathbf{F}(x, y) = \langle e^{y^2} + x, \ln(x^2 + 1) + 2y \rangle$. Use Green's theorem to find the normal component of $\mathbf{F}$ along C; i.e., find $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$, where $\mathbf{n}$ is the outward pointing normal vector for C.

`Using the divergence form of Green's theorem, we have that`

$$\int_C (\mathbf{F} \cdot \mathbf{n})\, ds = \iint_R \texttt{div}\, \mathbf{F}\, dA = 3 \iint_R dA = 3(\texttt{area}(R)) = 6\pi.$$

6. Let $\mathbf{F}(x, y, z) = \langle 1, x + yz, xy - \sqrt{z} \rangle$. Calculate $\operatorname{curl} \mathbf{F}$.

$$\texttt{curl}\, \mathbf{F} = \langle x - y, -y, 1 \rangle.$$

7. (a) Express the area of the surface $\mathbf{r}(u, v) = \langle uv, u + v, u - v \rangle$, with domain D described by $u^3 \leq v \leq u^2$, $0 \leq u \leq 1$, as a double integral in u and v. Do not evaluate the integral.

`Note that`

$$\mathbf{r}_u = \langle v, 1, 1 \rangle \qquad \texttt{and} \qquad \mathbf{r}_v = \langle u, 1, -1 \rangle,$$

`so that`

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{(-2)^2 + (v + u)^2 + (v - u)^2} = \sqrt{4 + 2u^2 + 2v^2}.$$

`Thus we have`

$$A = \int_0^1 \int_{u^3}^{u^2} \sqrt{4 + 2u^2 + 2v^2}\, dv\, du.$$

(b) Find the upward flux of the vector field $\mathbf{F}(x, y, z) = \langle y, x, z \rangle$ across the surface

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, 1 - u^2 \rangle,$$

where the domain D in the uv-plane is the region with $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

`We calculate`

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v, \sin v, -2u \rangle \times \langle -u \sin v, u \cos v, 0 \rangle = \langle 2u^2 \cos v, 2u^2 \sin v, u \rangle,$$

`while`

$$\mathbf{F}(\mathbf{r}(u, v)) = \langle u \sin v, u \cos v, 1 - u^2 \rangle.$$

`So we have`

$$F = \int_0^{2\pi} \int_0^1 \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA = \int_0^{2\pi} \int_0^1 (4u^3 \sin v \cos v + u - u^3)\, du\, dv = \frac{\pi}{2}.$$

Bonus: Describe this surface (verbally or visually).

`This is a paraboloid based at` $(0, 0, 1)$ `opening downward.`