

Name:

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Instructions: No calculators allowed. For full credit be sure to show all of your work and indicate your answer clearly. Unjustified answers will not receive any credit. Each problem is worth a total of 36 points.

1. Consider the linear map $L: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ given by $L(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 11 & 9 \\ -3 & -12 & -10 \end{bmatrix}.$$

- Find all the eigenvalues of A .
- For each eigenvalue of A , find a basis for the corresponding eigenspace.
- Find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP = D$.
- Find the inverse of the matrix P in part (c).
- Verify directly that $P^{-1}AP = D$.
- Find the determinant of A and verify directly that the determinant is equal to the product of the eigenvalues.

2. Consider the linear map $L: P_2 \rightarrow P_1$ given by $L(p(t)) = p'(t)$. Let S and S' be the following two bases for P_2 :

$$S = \{t^2, t, 1\} \quad S' = \{t^2 + 1, t + 1, t - 1\},$$

and let T and T' be the following two bases for P_1 :

$$T = \{t, 1\} \quad T' = \{t + 1, t\}.$$

- (a) Can L be an isomorphism? Explain.
- (b) Find the S' -coordinates for $p(t) = 3t^2 + 2t + 6$.
- (c) Find the transition matrix from S' to S .
- (d) Find the transition matrix from T to T' .
- (e) Find the matrix representation for L with respect to S and T .
- (f) Use your answers from parts (c)-(e) above to find the matrix representation for L with respect to S' and T' .

3. (a) Show that the kernel of any linear map $L: V \rightarrow W$ is a subspace of V .

(b) Suppose $L: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is defined by

$$L \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a + 2b + c \\ 2a + 4b - 3c \\ a + 2b - c \end{bmatrix}.$$

Find a basis for the kernel of L .

(c) Find a basis for the range of L .

(d) Verify the rank-nullity theorem using your answers to parts (b) and (c).

(e) Use Gram-Schmidt on your basis from part (c) to find an orthonormal basis for the range of L .

(f) Define the *span* of two vectors \mathbf{v}_1 and \mathbf{v}_2 , and show that if a vector \mathbf{w} is orthogonal to both \mathbf{v}_1 and \mathbf{v}_2 , then it is orthogonal to every vector in their span.

4. For each of the following twelve statements, indicate clearly whether it is true or false. For SIX of the statements, also do the following: If the statement is true, then explain why. If the statement is false, then provide a counterexample.

- (a) _____ If \mathbf{x} is an eigenvector of A , then so is $k\mathbf{x}$ for any scalar k .
- (b) _____ If a matrix is diagonalizable, then it is invertible.
- (c) _____ If a 3×3 matrix has eigenvalues $\lambda = 2, 1, -3$, then A is diagonalizable.
- (d) _____ Any set of n distinct non-zero vectors in an n -dimensional vector space is a basis for that space.
- (e) _____ If B is the reduced row echelon form of A then $\det(B) = \det(A)$.
- (f) _____ If $L: V \rightarrow W$ is a linear transformation, then for any vector \mathbf{w} in W there is a vector \mathbf{v} in V so that $L(\mathbf{v}) = \mathbf{w}$.
- (g) _____ If a linear transformation $L: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is one-to-one, then it is invertible.
- (h) _____ If $L: V \rightarrow W$ is a linear transformation and $\dim(V) > \dim(W)$, then L is onto.
- (i) _____ If $L: P_2 \rightarrow P_2$ is a linear transformation with nontrivial kernel, then L is not onto.
- (j) _____ Every orthonormal set of vectors is linearly independent.
- (k) _____ If A is a square matrix and A^2 is the zero matrix, then so is A .
- (l) _____ If V and W are finite dimensional vector spaces with $\dim(V) = \dim(W)$, then V and W are isomorphic.