Linear Algebra I **Final Exam** **December 13, 2004**

Name: OUID:

Instructions: No calculators allowed. For full credit be sure to show all of your work and indicate your answer clearly. Unjustified answers will not receive any credit. Each problem is worth a total of 36 points.

1. Consider the linear map $L\colon \mathbf{R}^3 \to \mathbf{R}^3$ given by $L(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 11 & 9 \\ -3 & -12 & -10 \end{bmatrix}.$$

(a) Find all the eigenvalues of A.

We have

$$\det(A - \lambda I) = (-1-\lambda)[(11-\lambda)(-10-\lambda)+108] = (-1-\lambda)(\lambda^2-\lambda-2)$$
$$= (-1-\lambda)(\lambda-2)(\lambda+1),$$

so the eigenvalues are $\lambda = -1$ twice, and $\lambda = 2$ once.

(b) For each eigenvalue of A, find a basis for the corresponding eigenspace.

For $\lambda = 2$ we have

$$\left[\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ 3 & 9 & 9 & 0 \\ -3 & -12 & -12 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so that the general solution is $\begin{bmatrix} 0 \\ -r \\ r \end{bmatrix} = r\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

For $\lambda = -1$, we have

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 3 & 12 & 9 & 0 \\ -3 & -12 & -9 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so that the general solution is $\begin{bmatrix} -4s-3r \\ s \\ r \end{bmatrix} = r\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$.

(c) Find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP = D$.

From the information above, the matrices are

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} 0 & -3 & -4 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(d) Find the inverse of the matrix P in part (d).

$$\left[\begin{array}{ccc|ccc} 0 & -3 & -4 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -4 & -3 \\ 0 & 1 & 0 & 1 & 4 & 4 \\ 0 & 0 & 1 & -1 & -3 & -3 \end{array}\right]$$

(e) Verify directly that $P^{-1}AP = D$.

Just multiply the above matrices.

(f) Find the determinant of A and verify directly that the determinant is equal to the product of the eigenvalues.

$$\texttt{det}(A) = -1(-110 + 108) = 2 = (-1)(-1)(2).$$

2. Consider the linear map $L\colon P_2 \to P_1$ given by $L(p(t)) = p'(t)$. Let S and S' be the following two bases for P_2:

$$S = \{t^2, t, 1\} \qquad S' = \{t^2 + 1, t + 1, t - 1\},$$

and let T and T' be the following two bases for P_1:

$$T = \{t, 1\} \qquad T' = \{t + 1, t\}.$$

(a) Can L be an isomorphism? Explain.

No. Spaces with different dimensions cannot be isomorphic.

(b) Find the S'-coordinates for $p(t) = 3t^2 + 2t + 6$.

(c) Find the transition matrix from S' to S.

(d) Find the transition matrix from T to T'.

(e) Find the matrix representation for L with respect to S and T.

We first plug the S-basis vectors into L, obtaining

$$L(t^2) = 2t, \qquad L(t) = 1, \qquad L(1) = 0.$$

Now we find T-coordinates of our outputs. These are

$$[2t]_T = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad [1]_T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad [0]_T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus the matrix is $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

(f) Use your answers from parts (c)-(e) above to find the matrix representation for L with respect to S' and T'.

3. (a) Show that the kernel of any linear map $L: V \to W$ is a subspace of V.

(b) Suppose $L: \mathbf{R}^3 \to \mathbf{R}^3$ is defined by

$$L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a + 2b + c \\ 2a + 4b - 3c \\ a + 2b - c \end{bmatrix}.$$

Find a basis for the kernel of L.

Row-reducing the standard matrix representation for L, we have

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -3 \\ 1 & 2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Augmenting this matrix with a column of 0s, we see that the kernel consists of vectors of the form

$$\begin{bmatrix} -2r \\ r \\ 0 \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

(c) Find a basis for the range of L.

Because the initial 1s in the row-reduced form appear in columns one and three, the corresponding columns of the original matrix representation corresponding to a basis for the range, namely

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}.$$

(d) Verify the rank-nullity theorem using your answers to parts (b) and (c).

The dimension of the kernel is one, the dimension of the range is two, and they sum to three, which is the dimension of the domain.

(e) Use Gram-Schmidt on your basis from part (c) to find an orthonormal basis for the range of L.

We take $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ as our first vector. Our second vector is then the original second vector, minus its projection onto the first. We have

$$\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} - \frac{-6}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

So our orthogonal basis is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. To get an orthonormal basis, divide these by their lengths, obtaining

$$\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{11}} \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}.$$

(f) Define the *span* of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, and show that if a vector $\mathbf{w}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, then it is orthogonal to every vector in their span.

4. For each of the following twelve statements, indicate clearly whether it is true or false. For SIX of the statements, also do the following: If the statement is true, then explain why. If the statement is false, then provide a counterexample.

(a) ______ If $\mathbf{x}$ is an eigenvector of A, then so is $k\mathbf{x}$ for any scalar k.

Technically this is false, because when k is zero, you get the zero vector, which is not allowed to be an eigenvector. As long as $k \neq 0$, it's true, though.

(b) ______ If a matrix is diagonalizable, then it is invertible.

False. The zero matrix is diagonalizable but not invertible.

(c) ______ If a 3×3 matrix has eigenvalues $\lambda = 2, 1, -3$, then A is diagonalizable.

True. Three distinct eigenvalues for a three by three matrix means they all have multiplicity one, and so cannot be defective. So there must be a basis of eigenvectors.

(d) ______ Any set of n distinct non-zero vectors in an n-dimensional vector space is a basis for that space.

False. You could have multiples of one vector, for instance.

(e) ______ If B is the reduced row echelon form of A then $\det(B) = \det(A)$.

False. Row-reducing changes the determinant, typically.

(f) ______ If $L: V \to W$ is a linear transformation, then for any vector $\mathbf{w}$ in W there is a vector $\mathbf{v}$ in V so that $L(\mathbf{v}) = \mathbf{w}$.

False. This is only true if the map is onto.

(g) ______ If a linear transformation $L: \mathbf{R}^3 \to \mathbf{R}^3$ is one-to-one, then it is invertible.

True. Because the dimensions are the same on both sides, one-to-one implies onto implies invertible.

(h) ______ If $L: V \to W$ is a linear transformation and $\dim(V) > \dim(W)$, then L is onto.

False. Consider the zero map.

(i) ______ If $L: P_2 \to P_2$ is a linear transformation with nontrivial kernel, then L is not onto.

True. Because the dimensions are the same, not one-to-one implies not onto.

(j) ______ Every orthonormal set of vectors is linearly independent.

True. Orthonormal implies none of them is the zero vector, and they ``point in different directions.''

(k) ______ If A is a square matrix and A^2 is the zero matrix, then so is A.

False.

(l) ______ If V and W are finite dimensional vector spaces with $\dim(V) = \dim(W)$, then V and W are isomorphic.

True. All n-dimensional vector spaces are isomorphic to $\mathbf{R}^n$, and thus to one another.