Linear Algebra I   **Test 1 Solutions**   **February 15, 2007**

1. (a) Show that the set of vectors of the form $\begin{bmatrix} a \\ b \end{bmatrix}$, where a is an integer, is not a subspace of $\mathbf{R}^2$.

This set is not closed under scalar multiplication. For example, $\pi \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \pi \\ \pi \end{bmatrix}$, which does not have an integer in the top entry. (Note that this set is in fact closed under addition.)

(b) Show that the kernel of a linear map is a subspace of the domain.

Suppose $\mathbf{v}$ and $\mathbf{w}$ are in the kernel of L. For closure under addition, we need to show that $\mathbf{v}+\mathbf{w}$ is also in the kernel. In other words, we need to show that $L(\mathbf{v}+\mathbf{w}) = \mathbf{0}$. But because L is linear, we have that $L(\mathbf{v}+\mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w}) = \mathbf{0}+\mathbf{0} = \mathbf{0}$, where the second-to-last equality comes from the fact that $\mathbf{v}$ and $\mathbf{w}$ are in the kernel. For closure under scalar multiplication, we need to show that $\alpha\mathbf{v}$ is in the kernel of L as long as $\mathbf{v}$ is. For this we note that $L(\alpha\mathbf{v}) = \alpha L(\mathbf{v}) = \alpha\mathbf{0} = \mathbf{0}$, where again we have used linearity of L and the fact that $\mathbf{v}$ is in the kernel.

2. Consider the linear map $L : \mathbf{R}^5 \to \mathbf{R}^4$ given by $L\left(\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}\right) = \begin{bmatrix} a-b+3c-d+2e \\ a+2c-d+3e \\ 2a-b+5c-d+2e \\ -b+c-e \end{bmatrix}$.

First find the standard matrix representation for L. Then verify the rank-nullity theorem by finding bases for both the kernel and the range of the map.

To get the standard matrix representation, we apply L to the standard basis vectors and then line up the results. To this end, we have

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

so that the matrix is

$$\begin{bmatrix} 1 & -1 & 3 & -1 & 2 \\ 1 & 0 & 2 & -1 & 3 \\ 2 & -1 & 5 & -1 & 2 \\ 0 & -1 & 1 & 0 & -1 \end{bmatrix}.$$

This matrix row-reduces as follows:

$$\begin{bmatrix}1 & -1 & 3 & -1 & 2\\ 1 & 0 & 2 & -1 & 3\\ 2 & -1 & 5 & -1 & 2\\ 0 & -1 & 1 & 0 & -1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & -1 & 3 & -1 & 2\\ 0 & 1 & -1 & 0 & 1\\ 0 & 1 & -1 & 1 & -2\\ 0 & -1 & 1 & 0 & -1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & -1 & 3 & -1 & 2\\ 0 & 1 & -1 & 0 & 1\\ 0 & 0 & 0 & 1 & -3\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

To describe the kernel, we solve the corresponding homogeneous system (i.e., imagine a column of 0s on the right of the row-reduced matrix). If the unknowns are a, b, c, d, e, we back-substitute to find that

$$e = r \qquad d = 3e = 3r \qquad c = s \qquad b = c - e = s - r \qquad a = b - 3c + d - 2e = -2s.$$

Thus vectors in the kernel look like

$$\begin{bmatrix}-2s\\ s-r\\ s\\ 3r\\ r\end{bmatrix} = r\begin{bmatrix}0\\ -1\\ 0\\ 3\\ 1\end{bmatrix} + s\begin{bmatrix}-2\\ 1\\ 1\\ 0\\ 0\end{bmatrix}.$$

It follows that these two vectors span the kernel. They are visibly linearly independent, and therefore form a basis for the kernel. Thus we have basis

$$\left\{\begin{bmatrix}0\\ -1\\ 0\\ 3\\ 1\end{bmatrix}, \begin{bmatrix}-2\\ 1\\ 1\\ 0\\ 0\end{bmatrix}\right\},$$

from which it follows that the dimension of the kernel is two.

For the range, we use the fact that the columns of the original matrix span the range, so we have only to find a basis inside this spanning set. Because the matrix row-reduces to have leading terms in columns one, two, and four, it follows that the first, second, and fourth columns are form a basis for the column space, and thus for the range of L. So a basis for the range is

$$\left\{\begin{bmatrix}1\\ 1\\ 2\\ 0\end{bmatrix}, \begin{bmatrix}-1\\ 0\\ -1\\ -1\end{bmatrix}, \begin{bmatrix}-1\\ -1\\ -1\\ 0\end{bmatrix}\right\},$$

from which it follows that the dimension of the range is three. The dimension of the domain of L is five, and $3+2=5$, verifying the rank-nullity theorem.

3. (a) Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V and let $L : V \to W$ be a linear map. Show that the range of L (in W) is spanned by $T = \{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)\}$. (Hint: Your answer should begin as indicated below. After completing the first sentence, write out $\mathbf{v}$ in terms of the vectors in S. Then hit this expression with L and use linearity to write the result as a combination of vectors in T. Deduce the result.)

Answer: Suppose $\mathbf{w}$ is in the range of L. Then there is some vector $\mathbf{v}$ so that…

$\ldots L(\mathbf{v}) = \mathbf{w}$. Because S is a basis, we may write

$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$$

for some real numbers $a_1, \ldots, a_n$. But then using the linearity of L we have that

$$\mathbf{w} = L(\mathbf{v}) = L(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = a_1L(\mathbf{v}_1) + \cdots + a_nL(\mathbf{v}_n).$$

This presents $\mathbf{w}$ as a linear combination of the vectors in T. As $\mathbf{w}$ was an arbitrary vector in the range, it follows that T spans the range.

(b) Must T be linearly independent? If yes, prove it; if no, give an explicit counterexample.

No. Take the zero map from $\mathbf{R}^n$ to itself, for instance.

4. For each of the following seven statements, indicate clearly whether it is true or false. (Assume V is finite dimensional.)

(a) If $\dim V = n$, then any set with fewer than n vectors must be linearly independent. False.

(b) If a linear operator $L : V \to V$ is 1-1, then it must be onto. True.

(c) A linearly independent set is always a basis for the subspace it spans. True.

(d) If $\dim V = n$, then any set of n or more vectors must span V. False.

(e) Any two bases for V must contain the same number of vectors. True.

(f) Every set that spans V contains a basis for V. True.

(g) Suppose W is a subspace of V and let S be a basis for V. Then some subset of S is a basis for W. False.

5. Suppose a matrix A row-reduces to $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(a) What can you say about the linear map $L : \mathbf{R}^4 \to \mathbf{R}^3$ defined by $L(\mathbf{v}) = A\mathbf{v}$?

It is not one-to-one; it is onto.

(b) What can you say about the columns of A as a set of vectors in $\mathbf{R}^3$?

They are not linearly independent; they do span $\mathbf{R}^3$; the vectors in columns one, two, and four form a basis for $\mathbf{R}^3$.

6. For each of the following scenarios, state which of the following phrases applies to any linear map $L : V \to W$:

L can be 1-1; L must be 1-1; L can be onto; L must be onto.

(a) $\dim V \le \dim W$ L can be 1-1; L can be onto.
(b) $\dim V < \dim W$ L can be 1-1.
(c) $\dim V \ge \dim W$ L can be 1-1; L can be onto.
(d) $\dim V > \dim W$ L can be onto.