Linear Algebra I Test 2 Solutions March 15, 2007

1. Let $L : P_2 \to P_2$ be defined by $L(at^2 + bt + c) = (2a - c)t^2 + (a + b - c)t + (c)$.

(a) Find the matrix representation for L with respect to the basis (on both sides)

$$S = \{t + 1,\ t^2 + t,\ t^2 + t + 1\}.$$

We first plug our basis vectors into the function, obtaining the following:

$$L(t+1) = -t^2 + 1,$$
$$L(t^2 + t) = 2t^2 + 2t,$$
$$L(t^2 + t + 1) = t^2 + t + 1.$$

We now find the S-coordinate vectors of these outputs. For the first, we solve

$$-t^2+1 = \alpha(t+1)+\beta(t^2+t)+\gamma(t^2+t+1) = (\beta+\gamma)t^2+(\alpha+\beta+\gamma)t+(\alpha+\gamma).$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

so that

$$[L(t+1)]_S = [-t^2 + 1]_S = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

For the next two outputs, we could set up and solve a system, but notice that both outputs are simple multiples of sinle basis vectors, so that we can determine their coordinates by inspection to be

$$[L(t^2 + t)]_S = [2t^2 + 2t]_S = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

and

$$[L(t^2 + t + 1)]_S = [t^2 + t + 1]_S = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus the matrix representation is

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Suppose $[\mathbf{v}]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $L(\mathbf{v})$ in two ways: once by finding $\mathbf{v}$ and plugging directly into L, and once by doing a matrix multiplication to find $[L(\mathbf{v})]_S$ and then undoing the coordinates.

(1) Using the given coordinates for $\mathbf{v}$, we find that

$$\mathbf{v} = 1(t+1) + 2(t^2+t) + 3(t^2+t+1) = 5t^2 + 6t + 4,$$

so that, using the given formula for L directly, we have

$$L(\mathbf{v}) = 6t^2 + 7t + 4.$$

(2) Applying the matrix we found above, we have

$$[L(\mathbf{v})]_S = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}.$$

Using these coordinates to find $L(\mathbf{v})$, we have

$$L(\mathbf{v}) = 1(t+1) + 3(t^2+t) + 3(t^2+t+1) = 6t^2 + 7t + 4.$$

(c) The transition matrix $P_{S \leftarrow T}$ from another basis T to S is given by

$$P_{S \leftarrow T} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Find the transition matrix from S to T by inverting this matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right].$$

Thus the transition matrix from S to T is

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

(d) Use the matrices from parts (a) and (c) to find the matrix representation for L with respect to T.

The matrix representation of L with respect to T is

$$P^{-1}AP = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & -1 & 6 \\ 0 & -1 & 4 \end{bmatrix}.$$

2. State and verify the rank-nullity theorem for the linear transformation $L : M_{22} \to M_{22}$ defined by

$$L(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A - A \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(Hint: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a random vector in M_{22}; write out $L(A)$ explicitly.)

The rank-nullity theorem states that

$$\dim \ker L + \dim \operatorname{range} L = \dim V.$$

For A as above, we have that

$$L(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix}.$$

To find the kernel, we set

$$L(A) = \begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and solve, finding that $c = 0$ and $a = d$. Thus the kernel is all matrices of the form

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We deduce that the kernel of L is spanned by the matrices

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

Because these matrices are linearly independent, they form a basis for the kernel. In particular, the kernel has dimension 2.

For the range, we have that matrices in the range of L are of the form

$$\begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix} = a \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + d \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It follows that these three matrices span the range. As the first is just the negative of the third, we find that a basis for the range is formed by

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

In particular, the range is 2-dimensional. These dimensions sum to four, which is the dimension of M_{22}, verifying the rank-nullity theorem.

3. For each of the following eight statements, indicate clearly whether it is true or false. (Assume V is finite dimensional.) For TWO of the statements, also do the following (in the space at the bottom of the page): If the statement is true, then explain why. If the statement is false, then provide a counterexample.

(a) ______ The set of all continuous functions f so that $f''(3) = 0$ is a subspace of $C(-\infty, \infty)$.

True. This can be checked directly.

(b) ______ If L is a linear map, then the image of a linearly independent set of vectors is always linearly independent.

False. The zero map provides a counterexample.

(c) ______ Transition matrices are always invertible.

True. The inverse is the reverse transition.

(d) ______ A matrix is invertible if and only if it row-reduces to the identity matrix.

True. This is how the inverse is found.

(e) ______ If a linear map is 1–1, then its matrix representation cannot contain any zero rows.

False. To be 1-1, each column of the matrix must contain an initial term after row-reducing. A matrix that is taller than it is wide can have an initial term in every column and still have some zero rows.

(f) ______ If $\dim V = n$, then any set with fewer than n vectors must be linearly independent.

False. The set could contain multiples of the same vector, for instance.

(g) ______ If V is a 4-dimensional vector space, then V must be isomorphic to M_{22}.

True. Any 4-dimensional vector space is isomorphic to $\mathbf{R}^4$. In particular, M_{22} is isomorphic to $\mathbf{R}^4$, and so is isomorphic to any 4-dimensional vector space (if V is isomorphic to W, and W is isomorphic to U, then V is isomorphic to U).

(h) ______ Any two bases for V must contain the same number of vectors.

True. This is what makes dimension well-defined.