Linear Algebra I   Test 3 Solutions   April 19, 2007

1. Prove the Pythagorean theorem: If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 .$$

Note that

$$\|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{u}\rangle + 2\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle$$

The fact that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal means that $\langle \mathbf{u}, \mathbf{v}\rangle = 0$, so that the right side of the equation above becomes

$$\langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 .$$

2. Let V be the vector space of infinitely differentiable functions, and consider the linear operator $L : V \to V$ given by

$$L(f(t)) = t^2 f''(t) - 4tf'(t).$$

Verify that the function $f(t) = t - 3t^4$ is an eigenvector for this operator with corresponding eigenvalue $\lambda = -4$

We need to check that

$$L(t - 3t^4) = -4(t - 3t^4).$$

But

$$L(t-3t^4) = t^2(-36t^2) - 4t(1-12t^3) = -36t^4 - 4t + 48t^4 = 12t^4 - 4t = -4(t-3t^4).$$

3. Let W be the subspace of P_2 (with inner product $\langle p(t), q(t)\rangle = \int_0^1 p(t)q(t)\,dt$) spanned by the orthogonal vectors $p(t) = 1$ and $q(t) = 2t - 1$.

(a) Find a basis for $W^\perp$.

The vectors in $W^\perp$ are those that are perpendicular to a basis for W. Thus the vector $at^2 + bt + c$ is in $W^\perp$ precisely if

$$0 = \langle at^2 + bt + c, 1\rangle = \int_0^1 (at^2 + bt + c)\,dt = \frac{a}{3} + \frac{b}{2} + c,$$

and

$$0 = \langle at^2 + bt + c, 2t - 1\rangle = \int_0^1 (2at^3 + (b-a)t^2 + (2c-b)t - c)\,dt = \frac{a}{6} + \frac{b}{6}.$$

Solving the corresponding system, we obtain

$$\left[\begin{array}{ccc|c} \frac{1}{3} & \frac{1}{2} & 1 & 0 \\ \frac{1}{6} & \frac{1}{6} & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 3 & 6 & 0 \\ 0 & 1 & 6 & 0 \end{array}\right],$$

so the general solution is given by $c = r$, $b = -6r$, and $a = 6r$. Choosing $r = 1$, we obtain the basis $\{6t^2 - 6t + 1\}$. (See part (c) for an alternative way to obtain a basis.)

(b) Find the projection of t^2 onto W.

`The basis we are given is orthogonal, so we plug into the formula to obtain`

$$\texttt{proj}_W(t^2) = \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle t^2, 2t-1\rangle}{\langle 2t-1, 2t-1\rangle} (2t-1)$$

$$= \frac{\int_0^1 t^2\,dt}{\int_0^1 1\,dt} 1 + \frac{\int_0^1 (2t^3 - t^2)\,dt}{\int_0^1 (4t^2 - 4t + 1)\,dt} (2t-1) = \frac{1}{3} + \frac{1/6}{1/3}(2t-1) = t - \frac{1}{6}.$$

(c) Write t^2 as $p(t) + q(t)$, where $p(t)$ is in W and $q(t)$ is in $W^\perp$.

`The projection of` t^2 `onto` W `provides the` W `part of the expression. The` $W^\perp$ `part is then just` t^2 `minus the` W `part. So we have`

$$t^2 = (t - \frac{1}{6}) + (t^2 - t + \frac{1}{6}).$$

`Note that this automatically gives us a vector in` $W^\perp$`, namely` $t^2 - t + \frac{1}{6}$`. We know that` $W^\perp$ `is 1-dimensional (because` W `is 2-dimensional and` P_2 `is 3-dimensional), so this vector provides a basis for` $W^\perp$`.`

4. Find an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$, where

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -4 & -4 & 2 \\ -8 & -12 & 6 \end{bmatrix}.$$

(You do ***not*** have to invert P or multiply anything out; just find P and D.)

`For the eigenvalues, we compute the determinant of` $A - \lambda I$ `as follows:`

$$\texttt{det}(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ -4 & -4-\lambda & 2 \\ -8 & -12 & 6-\lambda \end{bmatrix} = -\lambda^3 + 4\lambda^2 - 4\lambda = -\lambda(\lambda - 2)^2.$$

`Thus the eigenvalues are` $\lambda = 0$ `(once) and` $\lambda = 2$ `(twice).`

`Now we find the eigenvectors. For` $\lambda = 0$`, we have`

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ -4 & -4 & 2 & 0 \\ -8 & -12 & 6 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

`so eigenvectors are of the form`

$$\begin{bmatrix} 0 \\ r/2 \\ r \end{bmatrix} = \frac{r}{2} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

For $\lambda = 2$, we have

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -4 & -6 & 2 & 0 \\ -8 & -12 & 4 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so eigenvectors are of the form

$$\begin{bmatrix} \frac{r}{2} - \frac{3s}{2} \\ s \\ r \end{bmatrix} = \frac{r}{2}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \frac{s}{2}\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}.$$

Thus we have, for instance, that

$$P = \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

5. For each of the following twelve statements, indicate clearly whether it is true or false. (Assume A and B are square matrices of the same size.) For TWO of the statements, also do the following (in the space at the bottom of the page): If the statement is true, then explain why. If the statement is false, then provide a counterexample.

(a) ______ The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is diagonalizable.

False. Because it's upper triangular, the eigenvalues are $\lambda = 0$ and $\lambda = 1$. The only eigenvalue that can be defective is $\lambda = 1$ (it's the only one with multiplicity greater than one). Subtracting 1 from the diagonal and row-reducing gives $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, whose null-space is only one-dimensional.

(b) ______ If $\det(A) = 0$, then A has at least two equal rows.

False. This is just one of many reasons a matrix may have determinant zero.

(c) ______ If B is invertible, then $\det(B^{-1}A) = \dfrac{\det(A)}{\det(B)}$

True. This follows from the facts that $\det(AB) = \det(A) + \det(B)$ and $\det(A^{-1}) = (\det(A))^{-1}$.

(d) ______ A is singular if and only if $\det(A) = 0$.

True. This is a theorem.

(e) ______ If B is the row-reduced echelon form of A, then $\det(A) = \det(B)$.

`False. All matrices with non-zero determinant row-reduce to the identity, which has determinant one.`

(f) ______ $\det(A + B) = \det(A) + \det(B)$.

`False. Most random matrices will fail to satisfy this equality.`

(g) ______ A matrix is diagonalizable if and only if it is nonsingular.

False. The zero matrix is singular but diagonalizable. The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is non-singular but non-diagonalizable.

(h) ______ Every 3×3 matrix has at least one real eigenvalue.

True. Eigenvalues are roots of polynomials. A 3×3 matrix has a cubic polynomial, which always has a real root (either because complex roots come in pairs, or because all degree-3 curves must cross the x-axis somewhere).

(i) ______ If all the eigenvalues of a matrix are real, then the matrix is diagonalizable.

False. An eigenvalue might be defective. Our standard non-diagonalizable matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has all real eigenvalues, for example.

(j) ______ Any two eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal.

`True. This is a theorem.`

(k) ______ Zero is not allowed to be an eigenvalue.

`False. The matrix in problem #3 has zero as an eigenvalue, for example. Eigenvectors are not allowed to be the zero vector, however.`

(l) ______ Upper triangular matrices are always diagonalizable.

False. Our standard non-diagonalizable example is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, for example. The matrix in (a) is also a counterexample.