

1. [3 pts] Suppose λ is an eigenvalue of the $n \times n$ matrix A . Show that the set of all eigenvectors associated with λ (along with the zero vector) is a subspace of \mathbf{R}^n . (This is called the *eigenspace* associated to λ .)

Suppose \mathbf{v} and \mathbf{w} are eigenvectors associated to λ . Then we have

$$A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \lambda\mathbf{v} + \lambda\mathbf{w} = \lambda(\mathbf{v} + \mathbf{w}),$$

so that $\mathbf{v} + \mathbf{w}$ is again an eigenvector associated to λ . Similarly,

$$A(\alpha\mathbf{v}) = \alpha A\mathbf{v} = \alpha\lambda\mathbf{v} = \lambda(\alpha\mathbf{v}),$$

so that $\alpha\mathbf{v}$ is again an eigenvector when $\alpha \neq 0$. When $\alpha = 0$, we obtain the zero vector, which is thrown in by assumption.

2. (a) [3 pts] Find the characteristic equation for the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$.

We calculate

$$\det \begin{bmatrix} 3 - \lambda & 0 & 0 \\ -2 & 3 - \lambda & -2 \\ 2 & 0 & 5 - \lambda \end{bmatrix} = (3 - \lambda)^2(5 - \lambda).$$

- (b) [3 pts] Find the eigenvalues for this matrix.

It follows from above that the eigenvalues for A are $\lambda = 3$ and $\lambda = 5$.

- (c) [3 pts] For each eigenvalue, find a basis for the corresponding eigenspace by solving the equation $A\mathbf{x} = \lambda\mathbf{x}$.

For $\lambda = 3$, we solve $A\mathbf{x} = 3\mathbf{x}$, or $(A - 3I)\mathbf{x} = \mathbf{0}$, which gives us

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 0 & -2 & 0 \\ 2 & 0 & 2 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],$$

so that the general solution is

$$\mathbf{v} = \begin{bmatrix} -r \\ s \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

These last two vectors thus form a basis for the eigenspace corresponding to the eigenvalue $\lambda = 3$.

For $\lambda = 5$, we solve $Ax = 5x$, or $(A - 5I)x = \mathbf{0}$, which gives us

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ -2 & -2 & -2 & 0 \\ 2 & 0 & 0 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],$$

so that the general solution is

$$\mathbf{v} = \begin{bmatrix} 0 \\ -r \\ r \end{bmatrix} = r \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

This last vector forms a basis for the eigenspace corresponding to the eigenvalue $\lambda = 5$.

(d) [3 pts] Explain why your answers above imply that A is diagonalizable.

We have one eigenspace of dimension 2 and one of dimension 1. Because they correspond to different eigenvalues, they are linearly independent. In particular, they all together span a three dimensional space. It follows they span all of \mathbf{R}^3 . Thus we have a basis of eigenvectors, so that A is diagonalizable.

(e) [3 pts] Verify directly that A is diagonalizable by showing that $P^{-1}AP = D$. (In particular, find P , P^{-1} , and D .)

We can do this in any order we want. For example, we can set

$$P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix},$$

so that

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Finding P^{-1} in the usual way, we check that

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D.$$

(f) [3 pts] The *Cayley-Hamilton theorem* says that every square matrix satisfies its own characteristic equation. Verify this theorem for A . In other words, if the characteristic polynomial is

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n,$$

show that

$$A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI_n = O.$$

We found that

$$p(\lambda) = (\lambda - 3)^2(\lambda - 5) = \lambda^3 - 11\lambda^2 + 39\lambda - 45,$$

so we calculate

$$\begin{aligned} & A^3 - 11A^2 + 39A - 45I \\ = & \begin{bmatrix} 27 & 0 & 0 \\ -98 & 27 & -98 \\ 98 & 0 & 125 \end{bmatrix} - 11 \begin{bmatrix} 9 & 0 & 0 \\ -16 & 9 & -16 \\ 16 & 0 & 25 \end{bmatrix} + 39 \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix} - 45 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ & = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Bonus: Suppose $C = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is a 2×2 symmetric matrix.

(a) [3 pts] Show that the eigenvalues of C are real by explicitly computing them.

We calculate

$$\det \begin{bmatrix} a - \lambda & b \\ b & c - \lambda \end{bmatrix} = \lambda^2 - (a + c)\lambda - b^2.$$

Solving using the quadratic equation, we find that

$$\lambda = \frac{a + c \pm \sqrt{(a + c)^2 - (4ac - b^2)}}{2} = \frac{a + c \pm \sqrt{(a - c)^2 + b^2}}{2}.$$

Because the quantity under the square root is the sum of two squares, it cannot be negative. Thus the roots are real.

(b) [3 pts] Show that C is diagonalizable by showing that either the eigenvalues are distinct, or C is already diagonal.

As long as the quantity under the radical is not zero, we obtain two distinct real roots. Thus the eigenvalues are distinct, and have linearly independent eigenvectors. These necessarily form a basis for \mathbf{R}^2 , so C is diagonalizable.

On the other hand, if the quantity under the square root is zero, this means in particular that $b = 0$ (and also that $a = c$, but that's not important here). In other words, C is diagonal.