

1. Compute e^A , where $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

We first compute eigenvalues, finding that

$$\det \begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3).$$

So the eigenvalues are $\lambda = 2$ and $\lambda = 3$. We now solve the system

$$\begin{cases} e^2 = 2a_1 + a_0 \\ e^3 = 3a_1 + a_0 \end{cases} \rightsquigarrow \begin{bmatrix} 2 & 1 & | & e^2 \\ 3 & 1 & | & e^3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 & | & e^2 \\ 0 & 1 & | & 3e^2 - 2e^3 \end{bmatrix},$$

so the solution is

$$a_0 = 3e^2 - 2e^3, \quad a_1 = e^3 - e^2.$$

It follows that

$$e^A = (e^3 - e^2) \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} + (3e^2 - 2e^3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2e^2 - e^3 & e^3 - e^2 \\ 2e^2 - 2e^3 & 2e^3 - e^2 \end{bmatrix}.$$

2. Suppose there are four teams in a curling league. At the end of the season, the results are as follows:

Team 1 beat teams 2 and 3, but lost to team 4.

Team 2 beat team 3, but lost to teams 1 and 4.

Team 3 beat team 4, but lost to teams 1 and 2.

Team 4 beat teams 1 and 2, but lost to team 3.

- (a) Form the corresponding matrix A that reflects these results, where

$$a_{ij} = \begin{cases} 1 & \text{if team } i \text{ beat team } j \\ 0 & \text{otherwise} \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

- (b) How small can the dominant eigenvalue for A be? How large? Explain.

The dominant eigenvalue is bounded between the maximal and minimal column sums, which are one and two.

(c) It turns out that the dominant eigenvalue is approximately 1.395, and the corre-

sponding eigenvector is $\mathbf{v} = \begin{bmatrix} 0.552 \\ 0.321 \\ 0.448 \\ 0.626 \end{bmatrix}$. How should the teams be ranked?

Team 4 is #1, team 1 is #2, team 3 is #3, and team 2 is #4.