Linear Algebra I **Worksheet 1 Solutions** **January 23, 2007**

1. [6 pts] Use Gaussian elimination to solve the following linear system:

$$\begin{aligned} x + y + 2z &= -1 \\ x \qquad + z &= 2 \\ 2x + y + 3z &= 1 \end{aligned}$$

Beginning with the corresponding augmented matrix, we perform the following elementary row reductions:

$$\begin{bmatrix} 1 & 1 & 2 & | & -1 \\ 1 & 0 & 1 & | & 2 \\ 2 & 1 & 3 & | & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & | & -1 \\ 0 & -1 & -1 & | & 3 \\ 0 & -1 & -1 & | & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & | & -1 \\ 0 & 1 & 1 & | & -3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Solving the corresponding linear system, we let $z = r$ so that then $y = -3 - z = -3 - r$, and $x = -1 - y - 2z = 2 - r$.

2. [6 pts] Find an equation relating a, b, and c so that the following linear system is consistent for any values of a, b, and c that satisfy that equation (note the relationship between this system and the one in the previous problem):

$$\begin{aligned} x + y + 2z &= a \\ x \qquad + z &= b \\ 2x + y + 3z &= c \end{aligned}$$

We perform the same sequence of row reductions as in the previous problem, but this time with a, b, and c on the right. The result is

$$\begin{bmatrix} 1 & 1 & 2 & | & a \\ 1 & 0 & 1 & | & b \\ 2 & 1 & 3 & | & c \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & | & a \\ 0 & -1 & -1 & | & b-a \\ 0 & -1 & -1 & | & c-2a \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & | & a \\ 0 & 1 & 1 & | & a-b \\ 0 & 0 & 0 & | & c-2a-(b-a) \end{bmatrix}.$$

From the third row, we see that for this system to have a solution, it must be that $c - 2a - (b - a) = 0$, or, in other words, that $c = a + b$.

3. [6 pts] Determine all values of a for which the resulting linear system has (a) no solution; (b) a unique solution; (c) infinitely many solutions:

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ x + y + (a^2 - 5)z &= a + 1 \end{aligned}$$

Row reducing gives the following:

$$\begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 1 & 2 & 1 & | & 3 \\ 1 & 1 & a^2-5 & | & a+1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & a^2-6 & | & a-1 \end{bmatrix}.$$

If $a^2-6=0$ but $a-1 \neq 0$, then the system will have no solution. This occurs when $a=\pm\sqrt{6}$. The system has infinitely many solutions when every entry in the last row is zero. This is not the case for any choice of a. Thus for every other value of a the system has a unique solution.

4. [3 pts] Is it possible for a system of three equations in two unknowns to be consistent? If so, give an explicit example; if not, explain why not.

Yes. A trivial example is $x=0; y=0; x+y=0$. More interesting examples can be constructed by beginning with any consistent system of two equations in two unknowns, and tacking on a third that is some multiple of the first plus some multiple of the second.