

1. [12 pts] Let $L : \mathbf{R}^4 \rightarrow \mathbf{R}^4$ be the linear transformation defined by

$$L \left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) = \begin{bmatrix} a - b \\ 0 \\ c - d \\ 0 \end{bmatrix}.$$

(a) Show directly that L is linear.

We first show that $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$. To this end we have

$$L \left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \right) = L \left(\begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{bmatrix} \right) = \begin{bmatrix} (a_1 + a_2) - (b_1 + b_2) \\ 0 \\ (c_1 + c_2) - (d_1 + d_2) \\ 0 \end{bmatrix}.$$

On the other hand, we have

$$L \left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} \right) + L \left(\begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \right) = \begin{bmatrix} a_1 - b_1 \\ 0 \\ c_1 - d_1 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 - b_2 \\ 0 \\ c_2 - d_2 \\ 0 \end{bmatrix} = \begin{bmatrix} (a_1 - b_1) + (a_2 - b_2) \\ 0 \\ (c_1 - d_1) + (c_2 - d_2) \\ 0 \end{bmatrix}.$$

These are equal, so the first property is verified.

For the second, note that

$$L \left(\alpha \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) = L \left(\begin{bmatrix} \alpha a \\ \alpha b \\ \alpha c \\ \alpha d \end{bmatrix} \right) = \begin{bmatrix} \alpha a - \alpha b \\ 0 \\ \alpha c - \alpha d \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} a - b \\ 0 \\ c - d \\ 0 \end{bmatrix} = \alpha L \left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right).$$

(b) Find the standard matrix representation for L .

We have that

$$\begin{aligned} L \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & L \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ L \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} & L \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}. \end{aligned}$$

It follows that the standard matrix representation for L is

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(c) Describe the kernel of L .

If $L(\mathbf{v}) = \mathbf{0}$, then $a - b = 0$ and $c - d = 0$. Thus the kernel of

L is the set of all vectors of the form $\begin{bmatrix} a \\ a \\ c \\ c \end{bmatrix}$.

(d) Describe the range of L .

The range is all vectors of the form $\begin{bmatrix} a - b \\ 0 \\ c - d \\ 0 \end{bmatrix}$, where a, b, c, d are arbitrary. Note that this is the same as the set of all vectors of the form $\begin{bmatrix} \alpha \\ 0 \\ \gamma \\ 0 \end{bmatrix}$.

2. [3 pts] Show that the set of vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $b = a + c$, forms a subspace of \mathbf{R}^3 .

We need to show that this set is closed under addition and scalar multiplication. For addition, suppose $\mathbf{v} = \begin{bmatrix} a_1 \\ a_1 + c_1 \\ c_1 \end{bmatrix}$ and $\mathbf{u} =$

$\begin{bmatrix} a_2 \\ a_2 + c_2 \\ c_2 \end{bmatrix}$ are two vectors in the set. We need to show that $\mathbf{v} + \mathbf{u}$ is also of the correct form to be in the set. But we have that

$$\mathbf{v} + \mathbf{u} = \begin{bmatrix} a_1 + a_2 \\ a_1 + c_1 + a_2 + c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ (a_1 + a_2) + (c_1 + c_2) \\ c_1 + c_2 \end{bmatrix},$$

which is of the correct form. This verifies closure under addition.

For scalar multiplication, we have that

$$\alpha \begin{bmatrix} a \\ a + c \\ c \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha(a + c) \\ \alpha c \end{bmatrix},$$

which verifies closure under scalar multiplication.

3. [6 pts] Given a finite set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in \mathbf{R}^n , show that $\text{span}(S)$ is a subspace of \mathbf{R}^n .

If \mathbf{v} and \mathbf{u} are in the span of S , that means we can write

$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k \quad \mathbf{u} = b_1\mathbf{v}_1 + \cdots + b_k\mathbf{v}_k$$

for some real numbers a_1, \dots, a_k and b_1, \dots, b_k . For closure under addition, note then that

$$\begin{aligned} \mathbf{v} + \mathbf{u} &= (a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k) + (b_1\mathbf{v}_1 + \cdots + b_k\mathbf{v}_k) \\ &= (a_1 + b_1)\mathbf{v}_1 + \cdots + (a_k + b_k)\mathbf{v}_k, \end{aligned}$$

which shows that $\mathbf{v} + \mathbf{u}$ is still in the span of S . Hence the span of S is closed under addition. (Note that we used the facts that vector addition is commutative, that scalar multiplication is associative, and that scalar multiplication distributes over vector addition.)

For closure under scalar multiplication, note that

$$\alpha\mathbf{v} = \alpha(a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k) = (\alpha a_1)\mathbf{v}_1 + \cdots + (\alpha a_k)\mathbf{v}_k,$$

which shows that $\alpha\mathbf{v}$ is still in the span of S . Hence the span of S is closed under scalar multiplication.