Linear Algebra I Worksheet 4 Solutions February 13, 2007

1. [3 pts] Do the following vectors span $\mathbf{R}^4$ (justify your answer):

$$\left\{ \begin{bmatrix}1\\0\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\0\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\-5\\2\end{bmatrix} \right\}$$

We want to know if for any a, b, c, d, there is always a solution to the system

$$a_1 \begin{bmatrix}1\\0\\2\\1\end{bmatrix} + a_2 \begin{bmatrix}0\\1\\-1\\2\end{bmatrix} + a_3 \begin{bmatrix}1\\0\\0\\2\end{bmatrix} + a_4 \begin{bmatrix}-1\\1\\-5\\2\end{bmatrix} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}.$$

To see this, we generally want to row reduce the whole augmented matrix. But because we are not interested in what the solution actually is, just whether there is one or not, we can simplify things by just row-reducing the first four columns. If we obtain a row of 0s, then there is no solution. Otherwise, there is. These four columns reduce as follows:

$$\begin{bmatrix}1&0&1&-1\\0&1&0&1\\2&-1&0&-5\\1&2&2&2\end{bmatrix} \rightsquigarrow \begin{bmatrix}1&0&1&-1\\0&1&0&1\\0&-1&-2&-3\\0&2&1&3\end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix}1&0&1&-1\\0&1&0&1\\0&0&-2&-2\\0&0&1&1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1&0&1&-1\\0&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}.$$

Because there is a row of 0s, the corresponding system cannot always have a solution. Thus the given set does not span $\mathbf{R}^4$.

2. [6 pts] (a) Find a basis for the kernel of the linear map $L : \mathbf{R}^4 \to \mathbf{R}^4$ corresponding to multiplication by the following matrix:

$$\begin{bmatrix}1&1&2&-1\\2&3&6&-2\\-2&1&2&2\\0&-2&-4&0\end{bmatrix}$$

To find the kernel, we line up the vectors in a matrix and augment with a column of 0s, then row reduce. Leaving out the 0s, we have

$$\begin{bmatrix}1&1&2&-1\\2&3&6&-2\\-2&1&2&2\\0&-2&-4&0\end{bmatrix} \rightsquigarrow \begin{bmatrix}1&1&2&-1\\0&1&2&0\\0&4&8&0\\0&1&2&0\end{bmatrix} \rightsquigarrow \begin{bmatrix}1&1&2&-1\\0&1&2&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

The corresponding homogeneous linear system has general solution gvein by

$$a_4 = r \qquad a_3 = s \qquad a_2 = -2a_3 = -2s \qquad a_1 = a_4 - 2a_3 - a_2 = r - 2s + 2s = r.$$

Thus an element of the kernel looks like

$$\begin{bmatrix} r \\ -2s \\ s \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

It follows that these two vectors span the kernel. In order to form a basis, we need to make this set of vectors linearly independent. But for a set with only two vectors in it, linear independence is the same thing as not being multiples of one another. Thus they are already linearly independent. It follows that a basis for the kernel is given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(b) Find a basis for the range of this same map.

Applythe map to a generic vector in $\mathbf{R}^4$, we have

$$L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 3 & 6 & -2 \\ -2 & 1 & 2 & 2 \\ 0 & -2 & -4 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a + b + 2c - d \\ 2a + 3b + 6c - 2a \\ -2a + b + 2c + 2d \\ -2b - 4c \end{bmatrix}$$

$$= a \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 3 \\ 1 \\ -2 \end{bmatrix} + c \begin{bmatrix} 2 \\ 6 \\ 2 \\ -4 \end{bmatrix} + d \begin{bmatrix} -1 \\ -2 \\ 2 \\ 0 \end{bmatrix}.$$

Thus these four vectors span the range of L. We now need to find a basis inside this spanning set. We row reduce (as before; note that these vectors are exactly the columns of the original matrix) to obtain

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Because the initial 1s are in columns one and two, it follows that the first and second vectors form a basis for the range.

The basis is thus
$$\left\{\begin{bmatrix}1\\2\\-2\\0\end{bmatrix},\begin{bmatrix}1\\3\\1\\-2\end{bmatrix}\right\}.$$

3. [6 pts] Show that the following is a basis for $\mathbf{R}^4$: $\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\1\end{bmatrix},\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\\1\end{bmatrix}\right\}$

We line them up and row reduce as follows:
$$\begin{bmatrix}1&0&1&0\\1&0&0&1\\0&1&0&1\\0&1&1&1\end{bmatrix}\rightsquigarrow\begin{bmatrix}1&0&1&0\\0&0&-1&1\\0&1&0&1\\0&0&1&0\end{bmatrix}\rightsquigarrow\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$
Here are a couple of ways of seeing then that this is in fact a basis for $\mathbf{R}^4$.

(1) If we imagine having started with a random vector in an augmented column, then we see that the corresponding reduced system will always have a solution (there's no row of 0s). It's linearly independent, because if we imagine having started with an augmented column of 0s, we see that the resulting system has a unique (trivial) solution (there's no column without an initial 1).

(2) The original vectors certainly span something. The fact that they row reduce to have initial 1s in every column means that this set is linearly independent. Thus they span a four-dimensional subspace of $\mathbf{R}^4$. The only one of these is $\mathbf{R}^4$ itself.

4. [3 pts] Find a basis for the subspace of $\mathbf{R}^4$ consisting of vectors of the form $\begin{bmatrix}a+c\\a-b\\b+c\\-a+b\end{bmatrix}$. What is its dimension?

This subspace consists of vectors of the form
$$\begin{bmatrix}a+c\\a-b\\b+c\\-a+b\end{bmatrix}=a\begin{bmatrix}1\\1\\0\\-1\end{bmatrix}+b\begin{bmatrix}0\\-1\\1\\1\end{bmatrix}+c\begin{bmatrix}1\\0\\1\\0\end{bmatrix}.$$

Thus these three vectors span the subspace. To find a basis, we line them up and row reduce, arriving at

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Because there are initial 1s in all three columns, it follows that these three vectors are linearly independent. They thus form a basis for the subspace they span. This subspace is three-dimensional.

5. [3 pts] Find a basis for $\mathbf{R}^4$ that includes the vectors $\begin{bmatrix} 2 \\ 3 \\ 6 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$.

We first augment this set of two vectors with the standard basis in order to form a spanning set for $\mathbf{R}^4$:

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Lining them up and row reducing, we obtain

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 0 \\ 6 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & 2 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The initial terms are in columns one, two, four, and six. Thus a basis is

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$