Linear Algebra I　　Worksheet 5 Solutions　　February 27, 2007

1. [9 pts] You are given the following vector and two ordered bases for $\mathbf{R}^3$:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \qquad \mathbf{v} = \begin{bmatrix} 7 \\ -1 \\ 9 \end{bmatrix}$$

(a) Find the coordinate vectors $[\mathbf{v}]_T$ and $[\mathbf{v}]_S$ directly.

`For` $[\mathbf{v}]_T$`, we have the following:`

$$\begin{bmatrix} 2 & 1 & 0 & | & 7 \\ 1 & 0 & 1 & | & -1 \\ 0 & 3 & 0 & | & 9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 & 0 & | & 7 \\ 0 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & -3 \end{bmatrix}$$

`which has solution`

$$[\mathbf{v}]_T = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}.$$

`For` $[\mathbf{v}]_S$`, we have the following:`

$$\begin{bmatrix} 1 & 0 & 0 & | & 7 \\ 0 & 1 & 1 & | & -1 \\ 1 & -2 & 3 & | & 9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & | & 7 \\ 0 & 1 & 1 & | & -1 \\ 0 & 0 & 5 & | & 0 \end{bmatrix}$$

`which has solution`

$$[\mathbf{v}]_S = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}.$$

(b) Find the transition matrix $P_{S\leftarrow T}$.

`We need to find the` S`-coordinates for each of the` T`-basis vectors. Solving all three systems simultaneously, (and going all the way to reduced row-echelon form for convenience) we have the following:`

$$\begin{bmatrix} 1 & 0 & 0 & | & 2 & | & 1 & | & 0 \\ 0 & 1 & 1 & | & 1 & | & 0 & | & 1 \\ 1 & -2 & 3 & | & 0 & | & 3 & | & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & | & 2 & | & 1 & | & 0 \\ 0 & 1 & 0 & | & 1 & | & -2/5 & | & 3/5 \\ 0 & 0 & 1 & | & 0 & | & 2/5 & | & 2/5 \end{bmatrix}$$

`It follows that`

$$P_{S\leftarrow T} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -2/5 & 3/5 \\ 0 & 2/5 & 2/5 \end{bmatrix}.$$

(c) Verify your answers to parts (a) and (b) by checking that

$$[\mathbf{v}]_S = P_{S \leftarrow T}[\mathbf{v}]_T.$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -2/5 & 3/5 \\ 0 & 2/5 & 2/5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$$

2. [3 pts] Let $T = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathbf{R}^2$. Given that $P_{S \leftarrow T} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$, find the basis S for $\mathbf{R}^2$.

`The columns of` $P_{S \leftarrow T}$ `are the` S`-coordinates of the` T`-basis vectors. Thus if we denote the` S`-basis by` $\{\mathbf{v}_1, \mathbf{v}_2\}$`, then we have`

$$2\mathbf{v}_1 - \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad 3\mathbf{v}_1 + 2\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

`Treating this as two equations in two unknowns (`$\mathbf{v}_1$ `and` $\mathbf{v}_2$`), we solve to find that`

$$\mathbf{v}_1 = \begin{bmatrix} 3/7 \\ 1/7 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1/7 \\ 2/7 \end{bmatrix}.$$

`Note that you also could have come up with four equations in four unknowns (two for each spot in each of the vectors).`

3. [3 pts] Let $S = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathbf{R}^2$. Given that $P_{S \leftarrow T} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, find the basis T for $\mathbf{R}^2$.

`This is easier than the previous one. The columns of` $P_{S \leftarrow T}$ `are precisely the` S`-coordinates of the` T`-basis vectors. Because we already have the` S`-basis, we just put the coordinates in front as coefficients to find the` T`-basis vectors. Thus we have`

$$T = \left\{ 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right\}.$$

4. [6 pts] Let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} 2a - c \\ a + b - c \\ c \end{bmatrix}$.

(a) Find the matrix representation for L with respect to the basis $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

We first plug the S-basis vectors into L, obtaining

$$L\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\\1\end{bmatrix},\quad L\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\0\\1\end{bmatrix},\quad L\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0\\1\end{bmatrix}.$$

Now we take these outputs and find their S-coordinate vectors. Note that the first and third are already S-vectors themselves, so we clearly have

$$\begin{bmatrix}1\\1\\1\end{bmatrix}_S=\begin{bmatrix}1\\0\\0\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}1\\0\\1\end{bmatrix}_S=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

For the second output vector, we find the coordinates by solving as follows:

$$\left[\begin{array}{ccc|c}1&0&1&-1\\1&1&0&0\\1&1&1&1\end{array}\right]\rightsquigarrow\left[\begin{array}{ccc|c}1&0&1&-1\\0&1&-1&1\\0&0&1&1\end{array}\right]$$

so that

$$\begin{bmatrix}-1\\0\\1\end{bmatrix}_S=\begin{bmatrix}-2\\2\\1\end{bmatrix}.$$

It follows that the matrix we seek is

$$\begin{bmatrix}1&-2&0\\0&2&0\\0&1&1\end{bmatrix}.$$

(b) Suppose $[\mathbf{v}]_S=\begin{bmatrix}1\\2\\1\end{bmatrix}$. Use your answer to part (a) to find $[L(\mathbf{v})]_S$ and $L(\mathbf{v})$.

Using the definition of what the matrix we found above actually does, we have

$$\begin{bmatrix}1&-2&0\\0&2&0\\0&1&1\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}-3\\4\\3\end{bmatrix}=[L(\mathbf{v})]_S,$$

so

$$L(\mathbf{v})=-3\begin{bmatrix}1\\1\\1\end{bmatrix}+4\begin{bmatrix}0\\1\\1\end{bmatrix}+3\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\1\\4\end{bmatrix}.$$

(By the way, $\mathbf{v}$ is $\begin{bmatrix}2\\3\\4\end{bmatrix}$.)