Linear Algebra I   Worksheet 6 Solutions   March 6, 2007

1. Consider the following bases for $\mathbf{R}^2$ and $\mathbf{R}^3$:

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \qquad S' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

$$T = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad T' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and note that the transition matrices are given by

$$P_{S \leftarrow S'} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \qquad Q_{T \leftarrow T'} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Let $L : \mathbf{R}^2 \to \mathbf{R}^3$ be a linear map whose matrix representation with respect to S and T is $A = \begin{bmatrix} -1 & -1 \\ 0 & 3 \\ 2 & -1 \end{bmatrix}$, and let $[\mathbf{v}]_{S'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

(a) [3 pts] Find the inverse of Q.

We have the following:

$$\left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right].$$

It follows that

$$Q^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 0 \end{bmatrix}.$$

Note that you could have found this by finding the transition matrix from T to T'.

(b) [3 pts] Use the matrices above and your answer to part (a) to find the matrix representation B for L with respect to S' and T'.

Using the fact that $B = Q^{-1}AP$, we have that

$$B = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 1 & 1 \end{bmatrix}.$$

(c) [3 pts] Use matrix multiplication to find $[\mathbf{v}]_S$, $[L(\mathbf{v})]_T$ and $[L(\mathbf{v})]_{T'}$

Using the fact that $[\mathbf{v}]_S = P_{S\leftarrow S'}[\mathbf{v}]_{S'}$, we have that

$$[\mathbf{v}]_S = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Using the fact that $[L(\mathbf{v})]_T = A[\mathbf{v}]_S$, we have that

$$[L(\mathbf{v})]_T = \begin{bmatrix} -1 & -1 \\ 0 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix}.$$

To find $[L(\mathbf{v}]_{T'}$, we can use either the fact that $[L(\mathbf{v}]_{T'} = B[\mathbf{v}]_{S'}$, or the fact that $[L(\mathbf{v})]_{T'} = Q^{-1}[L(\mathbf{v}]_T$. Either way, we find that

$$[L(\mathbf{v})]_{T'} = \begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix}.$$

(d) [3 pts] Find $\mathbf{v}$ and $L(\mathbf{v})$.

We know the S- and T-coordinates of $\mathbf{v}$ and $L(\mathbf{v})$, respectively (as well as the S'- and T'-coordinates), so we use this information to calculate

$$\mathbf{v} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad L(\mathbf{v}) = -3\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix}.$$

(e) [6 pts] Find the matrix representation for L with respect to the standard bases for $\mathbf{R}^2$ and $\mathbf{R}^3$.

We'll use the bases S and T, along with the matrix representation A given above. Now let S'' and T'' be the standard bases for $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively. Then we find that

$$P_{S\leftarrow S''} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad Q_{T''\leftarrow T} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(Note that we've already done Q in the correct direction.) Multiplying these, we find that the matrix representation C of L with respect to the standard bases S'' and T'' is

$$C = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 2 & -3 \\ 3 & -3 \end{bmatrix}.$$

2. [3 pts] Let V be the set of all ordered pairs (a,b) of real numbers with the operations

$$(a,b) \oplus (c,d) = (a+c, b+d) \qquad r \odot (a,b) = (a, rb).$$

Show that this set with these operations is *not* a vector space.

All eight identities are satisfied except #6:

$$(c+d) \odot \mathbf{v} = c \odot \mathbf{v} \oplus d \odot \mathbf{v}$$

for all real numbers c, d and vectors $\mathbf{v}$. To see that this does not hold, let $\mathbf{v} = (a,b)$, and note that the left-hand side is

$$(c+d) \odot \mathbf{v} = (c+d) \odot (a,b) = (a, (c+d)b) = (a, cb+db),$$

where the second equality comes from how we've defined the symbol $\odot$. On the other hand, the right-hand side of the purported identity is

$$c \odot \mathbf{v} \oplus d \odot \mathbf{v} = c \odot (a,b) \oplus d \odot (a,b) = (a, cb) \oplus (a, db) = (a+a, cb+db) = (2a, cb+db),$$

where the second equality comes again from the definition of the symbol $\odot$, while the third comes from how we've defined the symbol $\oplus$. Because the expression we obtained for the left side does not identically equal the expression we obtained for the right side, the identity does not hold when $\oplus$ and $\odot$ are defined in this way on this set of objects. Thus, this is not a vector space.