Linear Algebra I **Worksheet 7 Solutions** **March 13, 2007**

1. [12 pts] Consider the subset $S = \{t^2 + t,\ 2t^2 + 1,\ t + 2\}$ of vectors in P_2.

(a) Show directly that S is a basis.

We need to show that S spans P_2 and is linearly independent. To show it spans, let $\mathbf{v} = at^2 + bt + c$ be an arbitrary vector in P_2. We want to show that the following always has a solution:

$$\alpha(t^2 + t) + \beta(2t^2 + 1) + \gamma(t + 2) = at^2 + bt + c.$$

For this, we rewrite the left side as $(\alpha + 2\beta)t^2 + (\alpha + \gamma)t + (\beta + 2\gamma)$, and set corresponding coefficients equal. This leads to the system whose augmented matrix is as follows:

$$\begin{bmatrix} 1 & 2 & 0 & | & a \\ 1 & 0 & 1 & | & b \\ 0 & 1 & 2 & | & c \end{bmatrix}.$$

The left part of this matrix reduces to

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

It follows that the system above always has a solution, so S spanes P_2. To check that S is linearly independent, we solve the same system as above, but with $a = b = c = 0$. It's clear from the row-reduction already done that the resulting system has only the trivial solution, so that S is also linearly independent.

(b) Find $[\mathbf{v}]_S$, where $\mathbf{v} = t^2 + t - 5$.

We do this directly, using the above system with $a = 1$, $b = 1$, and $c = -5$. This leads to the following:

$$\begin{bmatrix} 1 & 2 & 0 & | & 1 \\ 1 & 0 & 1 & | & 1 \\ 0 & 1 & 2 & | & -5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & | & 1 \\ 0 & -2 & 1 & | & 0 \\ 0 & 1 & 2 & | & -5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & | & 1 \\ 0 & -2 & 1 & | & 0 \\ 0 & 0 & 5 & | & -10 \end{bmatrix}$$

from which we determine the solution $\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$.

(c) Find the transition matrix $P_{T \leftarrow S}$, where $T = \{t^2 + t,\ t + 1,\ 1\}$.

We row-reduce as follows:

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 & 2 & 0 \\ 1 & 1 & 0 & | & 1 & 0 & 1 \\ 0 & 1 & 1 & | & 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & | & 1 & 2 & 0 \\ 0 & 1 & 0 & | & 0 & -2 & 1 \\ 0 & 0 & 1 & | & 0 & 3 & 1 \end{bmatrix},$$

so that $P_{T \leftarrow S} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$.

(d) If $[\mathbf{u}]_T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, what is $\mathbf{u}$?

$\mathbf{u} = 1(t^2 + t) + 2(t + 1) + 3(1) = t^2 + 3t + 5$.

2. [6 pts] Consider the function $L : P_2 \to \mathbf{R}^1$ given by $L(p(t)) = \int_{-1}^{1} p(t)\,dt$.

(a) Show that L is linear.

Let $p(t)$ and $q(t)$ be vectors in P_2. Then

$$L(p+q) = \int_{-1}^{1} (p+q)(t)\,dt = \int_{-1}^{1} p(t)\,dt + \int_{-1}^{1} q(t)\,dt = L(p) + L(q),$$

and

$$L(\alpha p) = \int_{-1}^{1} (\alpha p)(t)\,dt = \alpha \int_{-1}^{1} p(t)\,dt = \alpha L(p).$$

(b) Find the kernel of L. What is its dimension?

Suppose $L(p(t)) = 0$, and write $p(t) = at^2 + bt + c$ for some a, b, c. Then

$$0 = \int_{-1}^{1} (at^2 + bt + c)\,dt = 2a/3 + 2c,$$

so the kernel is all polynomials of the form at^2+bt+c, where $2a/3 + 2c = 0$, or where $a = -3c$. This is 2--dimensional (one way to see this is using the rank-nullity theorem: the domain has dimension 3, while the range clearly has dimension 1; thus the kernel has dimension 2).

3. [3 pts] Let W be the subspace of M_{22} spanned by the following matrices:

$$\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \right\}$$

Use an isomorphism between M_{22} and $\mathbf{R}^4$ to find a basis for W inside this set.

Asking for a basis inside this set is equivalent to asking for a basis inside the set of columns:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Here we're turning 2×2 matrices into columns using the isomorphism which takes coordinates with respect to the basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

so that the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ becomes the column $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. Lining up the columns above and then row-reducing, we obtain

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It follows that all four columns are linearly independent, and so are a basis unto themselves. Thus the original four matrices are independent and form a basis for W.