

Linear Algebra I Worksheet 8 Solutions April 3, 2007

1. Let V be the vector space of continuous functions on $[0, 1]$ and define an inner product by $\langle f, g \rangle = \int_0^1 f(t)g(t) dt$.

(a) Find the cosine of the angle between t^2 and t .

$$\cos \theta = \frac{\langle t^2, t \rangle}{\|t^2\| \|t\|} = \frac{\int_0^1 t^3 dt}{\left(\int_0^1 t^4 dt\right)^{1/2} \left(\int_0^1 t^2 dt\right)^{1/2}} = \frac{\sqrt{15}}{4}.$$

(b) Find the length of t^2 .

From above, we have that $\|t^2\| = 1/\sqrt{5}$.

(c) For what values of a and b are $3t + 1$ and $at + b$ orthogonal?

For the inner product, we have

$$\langle 3t + 1, at + b \rangle = \int_0^1 (3at^2 + (a + 3b)t + b) dt = a + \frac{a + 3b}{2} + b.$$

Setting this equal to zero, we obtain

$$2a + a + 3b + 2b = 3a + 5b = 0.$$

Any a and b satisfying this relationship produce a vector orthogonal to $3t + 1$.

(d) Let W be the subspace of V spanned by $\{1, 2t - 1, 6t^2 - 6t + 1\}$. Note that this is an orthogonal basis for W . Find the vector in W nearest to the vector given by $f(t) = \cos(t)$.

The vector in W nearest $\cos t$ is the projection of $\cos t$ onto W . Because the given basis is orthogonal, we use the formula

$$\begin{aligned} \text{proj}_W \cos t &= \frac{\langle \cos t, 1 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle \cos t, 2t - 1 \rangle}{\langle 2t - 1, 2t - 1 \rangle} (2t - 1) \\ &\quad + \frac{\langle \cos t, 6t^2 - 6t + 1 \rangle}{\langle 6t^2 - 6t + 1, 6t^2 - 6t + 1 \rangle} (6t^2 - 6t + 1). \end{aligned}$$

Evaluating all these integrals, we obtain

$$\begin{aligned} \text{proj}_W \cos t &= \frac{\sin(1)}{1} 1 + \frac{2 \cos(1) + \sin(1) - 2}{1/3} (2t - 1) \\ &\quad + \frac{6 \cos(1) - 11 \sin(1) + 6}{1/5} (6t^2 - 6t + 1) = \end{aligned}$$

$$[180 \cos(1) - 330 \sin(1) + 180]t^2 + [-168 \cos(1) + 336 \sin(1) - 192]t + [24 \cos(1) - 57 \sin(1) + 36].$$

2. Prove the parallelogram law for any two vectors in an inner product space:

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

$$\begin{aligned} & \|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 \\ &= \langle u + v, u + v \rangle + \langle u - v, u - v \rangle \\ &= (\langle u, u \rangle + 2\langle u, v \rangle + \langle v, v \rangle) + (\langle u, u \rangle - 2\langle u, v \rangle + \langle v, v \rangle) \\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2. \end{aligned}$$

3. Let W be the subspace of \mathbf{R}^4 spanned by $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}$.

(a) Find a basis for W^\perp .

A vector is in W^\perp if and only if its inner product with each of the basis vectors for W is zero. Thus we solve

$$\left\langle \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\rangle = 0, \quad \left\langle \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right\rangle = 0.$$

This gives a system of two equations in four unknowns, from which we obtain the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \end{array} \right].$$

This matrix is already row-reduced, so we see a general solution is

$$\begin{bmatrix} -2s - r \\ 2r - s \\ s \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

These two vectors form a basis for W^\perp .

(b) Write $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ as the sum of a vector in W and a vector in W^\perp .

For this, we first project the vector onto W , obtaining

$$\begin{aligned} \text{proj}_W \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} &= \frac{\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \frac{\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right\rangle} \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix} \\ &= \frac{11}{6} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \frac{-3}{6} \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 11 \\ -3 \\ 19 \\ 17 \end{bmatrix}. \end{aligned}$$

To find the W^\perp -component, we simply subtract:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 11 \\ -3 \\ 19 \\ 17 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -5 \\ 15 \\ -1 \\ 7 \end{bmatrix}.$$

Thus we have the decomposition

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 11 \\ -3 \\ 19 \\ 17 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} -5 \\ 15 \\ -1 \\ 7 \end{bmatrix}.$$