Linear Algebra I   Worksheet 9 Solutions   April 10, 2007

1. [6 pts] Find the Fourier polynomial of degree 2 for $\sin^2(t)$.

The formula gives

$$\sin^2(t) \approx$$

$$\frac{\int_{-\pi}^{\pi} \sin^2(t)\,dt}{\int_{-\pi}^{\pi} 1\,dt} 1 + \frac{\int_{-\pi}^{\pi} \sin^2(t)\cos(t)\,dt}{\int_{-\pi}^{\pi} \cos^2(t)\,dt} \cos(t) + \frac{\int_{-\pi}^{\pi} \sin^3(t)\,dt}{\int_{-\pi}^{\pi} \sin^2(t)\,dt} \sin(t)$$

$$+\frac{\int_{-\pi}^{\pi} \sin^2(t)\cos(2t)\,dt}{\int_{-\pi}^{\pi} \cos^2(2t)\,dt} \cos(2t) + \frac{\int_{-\pi}^{\pi} \sin^2(t)\sin(2t)\,dt}{\int_{-\pi}^{\pi} \sin^2(2t)\,dt} \sin(2t)$$

$$= \frac{\pi}{2\pi} 1 + \frac{0}{\pi}\cos(t) + \frac{0}{\pi}\sin(t) + \frac{\pi/2}{\pi}\cos(2t) + \frac{0}{\pi}\sin(2t)$$

$$= \frac{1}{2} - \frac{1}{2}\cos(2t).$$

Note that this is not just an approximation; it's an equality (half-angle formula). This is because $\sin^2(t)$ is actually contained in span$\{1, \cos(t), \sin(t), \cos(2t), \sin(2t)\}$, so when you project, the vector isn't moving, it's just being expressed differently, as a linear combination of the given basis vectors.

2. [6 pts] Use the Gram-Schmidt process to find an orthogonal basis for the kernel of the linear map $L : \mathbf{R}^4 \to \mathbf{R}^1$ defined by

$$L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = a - b - 2c + d.$$

We first find a basis for the kernel. Setting $a - b - 2c + d = 0$, we find a general solution is of the form

$$\begin{bmatrix} t + 2s - r \\ t \\ s \\ r \end{bmatrix} = r\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus these three vectors form a basis for the kernel. We now apply Gram-Schmidt to orthogonalize this basis.

For this we set $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. We now have

$$\mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{-2}{2} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

For the third orthogonal vector, we project $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ onto the span of $\mathbf{u}_1$ and $\mathbf{u}_2$, obtaining

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1 \\ -1/3 \\ 1/6 \end{bmatrix}.$$

(You should note that if you solve for the kernel differently, or put your basis in a different order, you will get a different answer.)

3. [6 pts] Find the matrix associated to the inner product $\langle p(t), q(t) \rangle = \int_{-1}^{1} p(t)q(t)\,dt$ on P_2, using the standard basis $\{1, t, t^2\}$.

$$\begin{bmatrix} \int_{-1}^{1} dt & \int_{-1}^{1} t\,dt & \int_{-1}^{1} t^2\,dt \\ \int_{-1}^{1} t\,dt & \int_{-1}^{1} t^2\,dt & \int_{-1}^{1} t^3\,dt \\ \int_{-1}^{1} t^2\,dt & \int_{-1}^{1} t^3\,dt & \int_{-1}^{1} t^4\,dt \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix}$$

4. [3 pts] Let $L_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear map which rotates counterclockwise about the origin by an angle θ, and let A_θ be the standard 2×2 matrix representation of L_θ. For which θ is A_θ diagonalizable? Explain. (You should never have to write out the matrix A_θ explicitly for this.)

Diagonalizable matrices always have a basis of eigenvectors. Eigenvectors are vectors that are only stretched by the matrix, not moved around.

Most rotations move vectors around. The only ones that send any vectors to parallel vectors are the trivial rotation (every non-zero vector is an eigenvector, and the only eigenvalue is 1) and rotation by 180° (every non-zero vector is an eigenvector and the only eignevalue is -1). So we get a diagonalizable matrix if and only if θ is an integer multiple of π.