INTRODUCTION

If D is an object, and f is a function defined at all points of D, then the integral of f over D does the following (with a few simplifications, which one can justify) :

- Chop D into small pieces, each of which has size ΔD
- For each piece D_i of D, pick a point p_i; then $f(p_i)$ is some number
- Add up the products $f(p_i)\Delta D$ for each i
- Take the limit as $\Delta D \to 0$, obtaining $\int_D f dD$

In this sense, all integrals do the same thing. When we say below that an integral is "adding up numbers along/across/over D," we mean to refer to this process as described above. The differences lie only in the notation, and in the most convenient way of actually calculating such a thing.

Notational conventions:

- We write as many $\int$ symbols as there are dimensions of D
- When D is one-dimensional (a curve C), we use Δs to denote a small piece of C ($\int_C f\,ds$); when C happens to be a piece of x-axis, we simply use Δx and omit the subscript on the integral ($\int f\,dx$)
- When D is two-dimensional (a surface S), we use ΔS to denote a small piece of S ($\iint_S f\,dS$); when S happens to be a piece of the xy-plane, we simply use ΔA ($\iint_D f\,dA$)
- When D is three-dimensional, it will (for us) automatically be a piece of xyz-space, and we use ΔV to denote a small piece of D ($\iiint_D f\,dV$)

INTEGRALS OVER CURVES C

An integral $\int_C f\,ds$ over a curve C simply adds up numbers along the curve, where the numbers are given by the function f. To calculate such an integral, we need to have a parametrization $\mathbf{r}(t)$ $(a \le t \le b)$ of the curve C. We then compute using the following

$$\int_C f\,ds = \int_a^b f\big(\mathbf{r}(t)\big)\,|\mathbf{r}'(t)|\,dt.$$

Special cases:

- When C is a piece of the x-axis, say from a to b, we use x as the parameter, and the expression above becomes

  $$\int_C f\,ds = \int_a^b f(x)\,dx.$$

  One often takes the function f to represent height (measured via the y-axis), so that this integral gives (signed) area under the graph $y = f(x)$.

- When the function f is identically 1, this integral simply adds up all the Δs bits, and hence gives the *arclength* of C. Thus we have

$$\text{Length}(C) = \int_C ds = \int_a^b |\mathbf{r}'(t)|\, dt.$$

- When C is a curve in the plane, note that we have

$$|\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

One often takes the function f to represent height (measured via the z-axis), so that this integral gives (signed) area of a curtain hanging from the curve $f(C)$ down to the xy-plane.

- When C is a curve in space, we have

$$|\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

- In the presence of a vector field $\mathbf{F}$ in the plane or space, we often consider the function f defined as the tangential component of $\mathbf{F}$ along C; i.e., $f = \mathbf{F} \cdot \mathbf{T}$, where T is the *unit* tangent vector $\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ to the curve C, so that the integral of this function along C becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \left(\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right) |\mathbf{r}'(t)|\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt,$$

where the left-most expression is just shorthand notation for the case where this particular function is the one being integrated. This is such a useful thing to do that we simply call it "integrating the vector field along the curve." It is important to remember that we're still just integrating a function over a curve. It's just that when we say that we are "integrating a vector field," we have a specific function in mind, namely, the one giving the tangential component of the vector field along the curve. When you write out this integral in terms of the components $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ of $\mathbf{F}$ and those of $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, you obtain the expression (in two dimensions, eliminate all terms with z or R)

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_C \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right) dt = \int_C P\, dx + Q\, dy + R\, dz,$$

where the right-most expression is just shorthand notation for the integral in the middle.

- When the vector field $\mathbf{F}$ is in the plane, we can also consider the function f defined as the normal component of $\mathbf{F}$ along C; i.e., $f = \mathbf{F} \cdot \mathbf{n}$, where n is the *unit* (outward-pointing) normal vector to the curve C, so that the integral of this function along C becomes

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b \left(\mathbf{F} \cdot \mathbf{n}\right) |\mathbf{r}'(t)|\, dt.$$

This is often called the "flux" of the vector field across the curve. When you write out this integral in terms of the components of $\mathbf{F}$ and of $\mathbf{r}$, you obtain the expression

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C \left(P\frac{dy}{dt} - Q\frac{dx}{dt}\right) dt = \int_C P\, dy - Q\, dx,$$

where, again, the right-most expression is just shorthand notation for the integral in the middle. (This doesn't make sense for curves in space, because there are lots of normal vectors to a curve in space; see the next section for 3D flux.)