

Sample problems for the final

1. A wire in space traces out the curve described by $\mathbf{r}(t) = \langle t^2, t^3, t^2 \rangle$ for $1 \leq t \leq 2\pi$. Find the total mass of the wire, given that its density at the point (x, y, z) is given by

$$f(x, y, z) = \sqrt{8 \left(\frac{xz}{y} \right)^2 + 9xz}.$$

2. (a) State the fundamental theorem for line integrals.

(b) Let C be the curve $\mathbf{r}(t) = \langle e^{t \sin(\pi t)}, t - t^5 \rangle$ for $0 \leq t \leq 1$. Verify that

$$\int_C (2e^{2x} \sin y - e^{x^2}) dx + (e^{2x} \cos y + \ln(y^2 + 1)) dy = 0,$$

by showing that this is the integral of a conservative vector field along a closed loop. *Do not find a potential function for the vector field.*

3. (a) Find a potential function for $\mathbf{F}(x, y) = \langle 2e^y \cos(2x) + 2xy, e^y \sin(2x) + x^2 \rangle$.

(b) Let C be the curve $\mathbf{r}(t) = \langle t \cos^2 t, e^{t^2 \sin t} \rangle$ for $0 \leq t \leq \pi$, and let \mathbf{F} be the vector field from part (a) above. Use your answer to part (a) to find the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ without directly calculating the integral.

4. Let C be the portion of the graph of $y = x^2 + x - 2$ where $0 \leq x \leq 1$, and let $\mathbf{F}(x, y) = \langle y - x, x^2 \rangle$. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

5. Let $\mathbf{F}(x, y, z) = \langle 1, x + yz, xy - \sqrt{z} \rangle$. Calculate $\operatorname{div} \mathbf{F}$ and $\operatorname{curl} \mathbf{F}$.

6. Express the area of the surface $\mathbf{r}(u, v) = \langle uv, u + v, u - v \rangle$, with domain D described by $u^3 \leq v \leq u^2$, $0 \leq u \leq 1$, as a double integral in u and v . Do not evaluate the integral.

7. Find the upward flux of the vector field $\mathbf{F}(x, y, z) = \langle y, x, z \rangle$ across the surface

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, 1 - u^2 \rangle,$$

where the domain D in the uv -plane is the region with $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

8. Use Stokes' theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle -y^2, x, z^2 \rangle$ and C is the intersection of the plane $y + z = 2$ with the cylinder $x^2 + y^2 = 1$.

9. Use the divergence theorem to calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle xy, y^2 + e^{xz^2}, \sin(xy) \rangle$, and S is the surface bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.