Sample problem solutions

1. A wire in space traces out the curve described by $\mathbf{r}(t) = \langle t^2, t^3, t^2 \rangle$ for $1 \le t \le 2\pi$. Find the total mass of the wire, given that its density at the point (x, y, z) is given by

$$f(x, y, z) = \sqrt{8\left(\frac{xz}{y}\right)^2 + 9xz}.$$

We need to integrate f along C. Note that

$$f(\mathbf{r}(t)) = \sqrt{8\left(\frac{t^2t^2}{t^3}\right)^2 + 9t^2t^2} = \sqrt{8t^2 + 9t^4},$$

and

$$|\mathbf{r}'(t)| = |\langle 2t, 3t^2, 2t \rangle| = \sqrt{4t^2 + 9t^4 + 4t^2} = \sqrt{8t^2 + 9t^4}.$$

Thus we have

$$\int_C f\, ds = \int_1^{2\pi} \sqrt{8t^2 + 9t^4}\sqrt{8t^2 + 9t^4}\, dt = \int_1^{2\pi} (8t^2 + 9t^4)\, dt$$

$$= \frac{64\pi^3}{3} + \frac{288\pi^5}{5} - \frac{8}{3} - \frac{9}{5}.$$

2. (a) State the fundamental theorem for line integrals.

$$\int_C \nabla f\, d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

(b) Let C be the curve $\mathbf{r}(t) = \langle e^{t\sin(\pi t)}, t - t^5 \rangle$ for $0 \le t \le 1$. Verify that

$$\int_C (2e^{2x}\sin y - e^{x^2})\, dx + (e^{2x}\cos y + \ln(y^2 + 1))\, dy = 0,$$

by showing that this is the integral of a conservative vector field along a closed loop. *Do not find a potential function for the vector field.*

The vector field is conservative because

$$\frac{\partial Q}{\partial x} = 2e^{2x}\cos y = \frac{\partial P}{\partial y},$$

and $\mathbf{F}$ is defined everywhere. The curve is closed because

$$\mathbf{r}(1) = \langle 1, 0 \rangle = \mathbf{r}(0).$$

It follows from the FTLI (or Green's theorem) that the integral is zero.

3. (a) Find a potential function for $\mathbf{F}(x,y) = \langle 2e^y \cos(2x) + 2xy, e^y \sin(2x) + x^2 \rangle$.

If $f(x,y)$ is a potential function, then

$$\frac{\partial f}{\partial x} = 2e^y \cos(2x) + 2xy,$$

which implies that

$$f(x,y) = e^y \sin(2x) + x^2 y + g(y),$$

for some function g of y. Then we have

$$\frac{\partial f}{\partial y} = e^y \sin(2x) + x^2 + g'(y),$$

from which it follows that $g(y) = 0$, so that

$$f(x,y) = e^y \sin(2x) + x^2 y.$$

(b) Let C be the curve $\mathbf{r}(t) = \langle t\cos^2 t, e^{t^2 \sin t} \rangle$ for $0 \le t \le \pi$, and let $\mathbf{F}$ be the vector field from part (a) above. Use your answer to part (a) to find the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ without directly calculating the integral.

Note that $\mathbf{r}(pi) = \langle \pi, 1 \rangle$ and $\mathbf{r}(0) = \langle 0, 1 \rangle$. By the FTLI we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = f(\pi, 1) - f(0,1) = \pi^2 - 0 = \pi^2.$$

4. Let C be the portion of the graph of $y = x^2 + x - 2$ where $0 \le x \le 1$, and let $\mathbf{F}(x,y) = \langle y - x, x^2 \rangle$. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

The curve C is not closed, and the vector field is not conservative, so we have no choice but to integrate directly. Thus using x as the parameter (so $\mathbf{r}(t) = \langle t, t^2 + t - 2 \rangle$) we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle t^2 + t - 2 - t, t^2 \rangle \cdot \langle 1, 2t + 1 \rangle \, dt$$

$$= \int_0^1 2t^3 + 2t^2 - 2 \, dt = \frac{1}{2} + \frac{2}{3} - 2 = -\frac{5}{6}.$$

5. Let $\mathbf{F}(x,y,z) = \langle 1, x + yz, xy - \sqrt{z} \rangle$. Calculate div$\mathbf{F}$ and curl$\mathbf{F}$.

$$\text{div}\,\mathbf{F} = z - \frac{1}{2\sqrt{z}},$$

$$\text{curl}\,\mathbf{F} = \langle x - y, -y, 1 \rangle.$$

6. Express the area of the surface $\mathbf{r}(u, v) = \langle uv, u+v, u-v \rangle$, with domain D described by $u^3 \leq v \leq u^2$, $0 \leq u \leq 1$, as a double integral in u and v. Do not evaluate the integral.

$$\texttt{Area}(S) = \iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

We compute that

$$|\mathbf{r}_u \times \mathbf{r}_v| = |\langle v, 1, 1\rangle \times \langle u, 1, -1\rangle|$$

$$= |\langle -2, v+u, v-u\rangle| = \sqrt{4 + (v+u)^2 + (v-u)^2} = \sqrt{2u^2 + 2v^2 + 4}.$$

Thus the area is

$$A = \int_0^1 \int_{u^3}^{u^2} \sqrt{2u^2 + 2v^2 + 4}\, dv\, du.$$

7. Find the upward flux of the vector field $\mathbf{F}(x, y, z) = \langle y, x, z\rangle$ across the surface

$$\mathbf{r}(u, v) = \langle u\cos v, u\sin v, 1 - u^2\rangle,$$

where the domain D in the uv-plane is the region with $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

We compute

$$\mathbf{F}(\mathbf{r}(u, v)) = \langle u\sin v, u\cos v, 1 - u^2\rangle,$$

and

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v, \sin v, -2u\rangle \times \langle -u\sin v, u\cos v, 0\rangle = \langle 2u^2\cos v, 2u^2\sin v, u\rangle.$$

We want the upward flux, so the z-coordinate of our normal vector needs to be positive. The z-coordinate of $\mathbf{r}_u \times \mathbf{r}_v$ is u, which for us is between 0 and 1, so this is the correct normal direction. So the integral is

$$\iint_S \mathbf{F}\, d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v,)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

$$= \int_0^1 \int_0^{2\pi} \langle u\sin v, u\cos v, 1 - u^2\rangle \cdot \langle 2u^2\cos v, 2u^2\sin v, u\rangle\, dv\, du$$

$$= \int_0^1 \int_0^{2\pi} (4u^3 \sin v\cos v + u - u^3)\, dv\, du = \int_0^1 (2\pi u - 2\pi u^3)\, du = \frac{\pi}{2}.$$

8. Use Stokes' theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle -y^2, x, z^2\rangle$ and C is the intersection of the plane $y + z = 2$ with the cylinder $x^2 + y^2 = 1$.

The curve C is closed, so that we can use Stokes' theorem with any surface it bounds. The simplest thing to do is to take S to be the piece of the plane $y+z=2$ that lies over the domain D, which is the region inside the unit circle in the xy-plane. We use x and y as parameters, so that $\mathbf{r}(u,v) = \langle u, v, 2-v \rangle$. Now we calculate

$$\texttt{curl}\,\mathbf{F} = \langle 0, 0, 1+2v \rangle$$

and

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 1, 0, 0 \rangle \times \langle 0, 1, -1 \rangle = \langle 0, 1, 1 \rangle.$$

Thus we calculate

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \texttt{curl}\,\mathbf{F} \cdot d\mathbf{S} = \iint_D \langle 0, 0, 1+2v \rangle \cdot \langle 0, 1, 1 \rangle \, dA = \iint_D (1+2v)\, dA.$$

To compute this double integral, we use polar coordinates, so we have

$$\iint_D (1+2v)\, dA = \int_0^{2\pi} \int_0^1 (1 + 2\sin\theta) r \, dr \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (1 + 2\sin\theta)\, d\theta = \pi.$$

9. Use the divergence theorem to calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = \langle xy, y^2 + e^{xz^2}, \sin(xy) \rangle$, and S is the surface bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.

This surface encloses a solid region, so we can use the divergence theorem. We compute

$$\texttt{div}\,\mathbf{F} = 3y.$$

Thus we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \texttt{div}\,\mathbf{F}\, dV = \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y \, dy \, dz \, dx = \frac{184}{35}.$$