**Calculus IV     Test 2 Solutions     March 9, 2006**

1. Consider the following:

$$\int_{-2}^{0}\int_{0}^{1+x/2} f(x,y)\,dy\,dx + \int_{0}^{1}\int_{0}^{1-x^2} f(x,y)\,dy\,dx.$$

(a) Carefully sketch the region of integration.

There's a straight line from the point $(-2,0)$ to the point $(0,1)$, and then a parabolic arc from $(0,1)$ to $(1,0)$.

(b) Express the sum as just one double integral (rather than as a sum of two) by reversing the order of integration.

$$\int_{0}^{1}\int_{2y-2}^{\sqrt{1-y}} f(x,y)\,dx\,dy$$

2. Express the following integral in spherical coordinates. It may help to sketch the solid region $E$ and/or the domain $D$. Do not evaluate the integral.

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z\,dz\,dx\,dy.$$

Using $\sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2}$, we see that the region lies between the standard sphere of radius $\sqrt{2}$ and the standard right circular cone. In the $xy$-plane, the fact that $-\sqrt{1-y^2} \le x \le 0$ tells us that the domain $D$ is a portion of the left half of the region inside the unit circle. Finally, because $-1 \le y \le 1$, we do in fact have the entire lieft side of the circle as our $D$.

This region is a spherical wedge. The fact that we are inside the sphere of radius $\sqrt{2}$ means that the bounds on $\rho$ are $0 \le \rho \le \sqrt{2}$. From the picture of $D$, we know that $\theta$ has $\pi/2 \le \theta \le 3\pi/2$. Finally, the right circular cone has declination $\pi/4$, so we know that $0 \le \phi \le \pi/4$. Using the fact that $z = \rho\cos\phi$, we have

$$\int_{0}^{\pi/4}\int_{\pi/2}^{3\pi/2}\int_{0}^{\sqrt{2}} (\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_{0}^{\pi/4}\int_{\pi/2}^{3\pi/2}\int_{0}^{\sqrt{2}} \rho^3\sin\phi\cos\phi\,d\rho\,d\theta\,d\phi.$$

3. A surface occupies the region $D$ in the $xy$-plane bounded by the curves $y = 2x^2$ and $y = x^3$. The temperature of the surface at the point $(x,y)$ is described by $T(x,y) = x+2$. Find the average temperature of the surface.

The region is both $x$- and $y$-simple. We will use the fact that it is $y$-simple, so we have that $x^3 \le y \le 2x^2$, for $0 \le x \le 2$. Thus we have

$$\int_0^2 \int_{x^3}^{2x^2} (x+2) dy\, dx = \int_0^2 (4x^2 - x^4)\, dx = \frac{32}{3} - \frac{32}{5}.$$

For the volume, we have

$$\int_0^2 \int_{x^3}^{2x^2} dy\, dx = \frac{16}{3} - \frac{16}{4}.$$

Thus the average temperature is

$$f_{\mathrm{avg}} = \frac{\frac{32}{3} - \frac{32}{5}}{\frac{16}{3} - \frac{16}{4}} = \frac{16}{5}.$$

4. The region $E$ is bounded by the parabolic cylinder $y = x^2$, the plane $z = y$, the plane $y = 4$, and the $xy$-plane. Express the integral $\iiint_E f(x,y,z)\, dV$ in two different ways, using $dV = dz\, dy\, dx$ and $dV = dx\, dy\, dz$.

As a $z$-simple region, lines in $E$ parallel to the $z$-axis start at the $xy$-plane $z = 0$ and end at the plane $z = y$. We thus have $0 \le z \le y$. The shadow of $E$ in the $xy$-plane is exactly the region above the parabola $y = x^2$ below $y = 4$. we thus have $x^2 \le y \le 4$ and $-2 \le x \le 2$. This gives

$$\iiint_E f(x,y,z)\, dV = \int_{-2}^2 \int_{x^2}^4 \int_0^y f(x,y,z)\, dz\, dy\, dx.$$

As an $x$-simple region, lines in $E$ parallel to the $x$-axis start on the back side of the parabolic cylinder $y = x^2$ and end on the front side. We thus have $-\sqrt{y} \le x \le \sqrt{y}$. The shadow of $E$ on the $yz$-plane is the triangle formed by the $y$-axis, the line $z = y$, and the line $y = 4$. We thus have $0 \le y \le z$ and $0 \le z \le 2$. This gives

$$\iiint_E f(x,y,z)\, dV = \int_0^4 \int_z^4 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\, dx\, dy\, dz.$$

5. A solid $E$ lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 1$, and below the paraboloid $z = 1 + x^2 + y^2$. Evaluate the integral

$$\iiint_E (x^2 + y^2)^{3/2}\, dV.$$

The region is $z$-simple, with $1 \le z \le 1 + x^2 + y^2$. The domain over which the object sits is the circle where the cylinder hits the $xy$-plane, so it's $x^2 + y^2 = 1$. This is most easily described in polar coordinates with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Thus we use cylindrical coordinates to calculate

$$\iiint_E (x^2 + y^2)^{3/2}\, dV = \int_0^{2\pi} \int_0^1 \int_1^{1+r^2} r^3\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r^6\, dr\, d\theta = \frac{1}{7} \int_0^{2\pi} d\theta = \frac{2\pi}{7}.$$