Test 1 Solutions

Calculus III **January 29, 2008**

1. Let $\mathbf{v} = \langle 1, 4, 2\rangle$ and $\mathbf{w} = \langle 1, 0, 1\rangle$, and let θ denote the angle between these two vectors. Compute the following:

(a) $2\mathbf{v} - 3\mathbf{w} =$

$$2\langle 1,4,2\rangle - 3\langle 1,0,1\rangle = \langle 2,8,4\rangle - \langle 3,0,3\rangle = \langle 2-3, 8-0, 4-3\rangle = \langle -1, 8, 1\rangle$$

(b) $\mathbf{v} \cdot \mathbf{w} =$

$$\langle 1,4,2\rangle \cdot \langle 1,0,1\rangle = (1)(1) + (4)(0) + (2)(1) = 1 + 0 + 2 = 3$$

(c) $\mathbf{v} \times \mathbf{w} =$

$$\langle 1,4,2\rangle \times \langle 1,0,1\rangle = \langle (4)(1) - (2)(0), (2)(1) - (1)(1), (1)(0) - (4)(1)\rangle = \langle 4, 1, -4\rangle$$

(d) $\text{proj}_{\mathbf{w}}(\mathbf{v}) =$

$$\frac{\langle 1,4,2\rangle \cdot \langle 1,0,1\rangle}{\langle 1,0,1\rangle \cdot \langle 1,0,1\rangle}\langle 1,0,1\rangle = \frac{3}{2}\langle 1,0,1\rangle = \langle 3/2, 0, 3/2\rangle$$

(e) $\cos(\theta) =$

$$\frac{\langle 1,4,2\rangle \cdot \langle 1,0,1\rangle}{|\langle 1,4,2\rangle||\langle 1,0,1\rangle|} = \frac{3}{\sqrt{1^2+4^2+2^2}\sqrt{1^2+0^2+1^2}} = \frac{3}{\sqrt{42}}$$

(f) $|\mathbf{v}| =$

$$\sqrt{1^2+4^2+2^2} = \sqrt{21}$$

2. Consider the helix described by $\mathbf{r} = \langle t, 2\cos(t), 2\sin(t)\rangle$. Compute each of the following at the point $(\frac{\pi}{4}, \sqrt{2}, \sqrt{2})$:

(a) velocity=

$$\mathbf{r}'(\pi/4) = \langle 1, -2\sin(\pi/4), 2\cos(\pi/4)\rangle = \langle 1, -\sqrt{2}, \sqrt{2}\rangle$$

(b) speed=

$$|\mathbf{r}'(\pi/4)| = \sqrt{1^2 + (-\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{5}$$

(c) acceleration=

$$\mathbf{r}''(\pi/4) = \langle 0, -2\cos(\pi/4), -2\sin(\pi/4)\rangle = \langle 0, -\sqrt{2}, -\sqrt{2}\rangle$$

(d) $\mathbf{T} =$

$$\frac{\mathbf{r}'(\pi/4)}{|\mathbf{r}'(\pi/4)|} = \tfrac{1}{\sqrt{5}}\langle 1, -\sqrt{2}, \sqrt{2}\rangle$$

(e) $\mathbf{N} =$

$$\frac{\mathbf{T}'(\pi/4)}{|\mathbf{T}'(\pi/4)|} = \frac{\frac{1}{\sqrt{5}}\langle 0, -2\cos(\pi/4), -2\sin(\pi/4)\rangle}{|\frac{1}{\sqrt{5}}\langle 0, -2\cos(\pi/4), -2\sin(\pi/4)\rangle|} =$$
$$\frac{1}{\sqrt{0^2 + (-\sqrt{2})^2 + (-\sqrt{2})^2}}\langle 0, -\sqrt{2}, -\sqrt{2}\rangle = \tfrac{1}{2}\langle 0, -\sqrt{2}, -\sqrt{2}\rangle$$

(f) $\mathbf{B} =$

$$\mathbf{T} \times \mathbf{N} = \tfrac{1}{\sqrt{5}}\tfrac{1}{2}\langle 1, -\sqrt{2}, \sqrt{2}\rangle \times \langle 0, -\sqrt{2}, -\sqrt{2}\rangle = \tfrac{1}{2\sqrt{5}}\langle 4, \sqrt{2}, -\sqrt{2}\rangle$$

(g) $\kappa =$

$$\frac{|\mathbf{T}'(\pi/4)|}{|\mathbf{r}'(\pi/4)|} = \frac{2/\sqrt{5}}{\sqrt{5}} = \frac{2}{5}$$