Test 2 Solutions

Calculus III **February 26, 2008**

1. Suppose x and y are the lengths of two sides of a triangle, and θ is the angle these two sides contain. There is a formula stating that the area of such a triangle is

$$A = \frac{1}{2}xy\sin\theta.$$

Now suppose your triangle is changing size over time, so that x, y, and θ are all functions of t.

(a) Write out the formula for the chain rule for $\frac{dA}{dt}$.

$$\frac{dA}{dt} = \frac{dA}{dx}\frac{dx}{dt} + \frac{dA}{dy}\frac{dy}{dt} + \frac{dA}{d\theta}\frac{d\theta}{dt}$$

(b) At time $t = 0$, you find that $x = 4$, $y = 6$, and $\theta = \pi/6$. Your measurements reveal that when $t = 0$, x is increasing at a rate of 2 in/s, y is decreasing at a rate of 3 in/s, and θ is increasing at a rate of $\frac{1}{20}$ rad/s. Use the chain rule to find the rate of change (per second) of the area of the triangle when $t = 0$.

We compute:

$$\frac{dA}{dx} = \frac{1}{2}y\sin\theta, \qquad \frac{dA}{dy} = \frac{1}{2}x\sin\theta, \qquad \frac{dA}{d\theta} = \frac{1}{2}xy\cos\theta.$$

Evaluating at the point $(4, 6, \pi/6)$, we find that

$$\frac{dA}{dx} = \frac{1}{2}(6)(1/2) = 3/2, \quad \frac{dA}{dy} = \frac{1}{2}(4)(1/2) = 1, \quad \frac{dA}{d\theta} = \frac{1}{2}(4)(6)(\sqrt{3}/2) = 6\sqrt{3}.$$

The derivatives with respect to t are provided in the problem, so we now compute

$$\left.\frac{dA}{dt}\right|_{t=0} = \frac{dA}{dx}\frac{dx}{dt} + \frac{dA}{dy}\frac{dy}{dt} + \frac{dA}{d\theta}\frac{d\theta}{dt} = (3/2)(2) + (1)(-3) + (6\sqrt{3})(1/20) = \frac{3\sqrt{3}}{10}.$$

2. Write an explicit system of equations that can be used to optimize $f(x, y) = x^2y$ subject to the constraint $x^2 + y^2 = 4$. Do not solve the system.

We compute gradients:

$$\nabla f = \langle 2xy, x^2 \rangle \qquad \nabla g = \langle 2x, 2y \rangle .$$

Using the method of Lagrange multipliers, we set $\nabla f = \lambda \nabla g$, and use the constraint equation, to obtain the system

$$\begin{cases} 2xy = 2\lambda x \\ x^2 = 2\lambda y \\ x^2 + y^2 = 4 \end{cases} .$$

3. Let $f(x,y) = x^2 + y^2 + kxy$, where k is a constant.

(a) Show that $(0,0)$ is always a critical point for f, regardless of the value of k.

We compute:

$$f_x(0,0) = (2x + ky)|_{(0,0)} = 0$$

and

$$f_y(0,0) = (2y + kx)|_{(0,0)} = 0.$$

Because both first partials are both zero at $(0,0)$, this is always a critical point for f, regardless of k.

(b) For which values of k is this critical point a local maximum? a local minimum? a saddle? For which values of k does the second derivative test fail?

The second derivatives are

$$f_{xx}(0,0) = 2, \qquad f_{yy}(0,0) = 2, \qquad f_{xy}(0,0) = k,$$

so we have

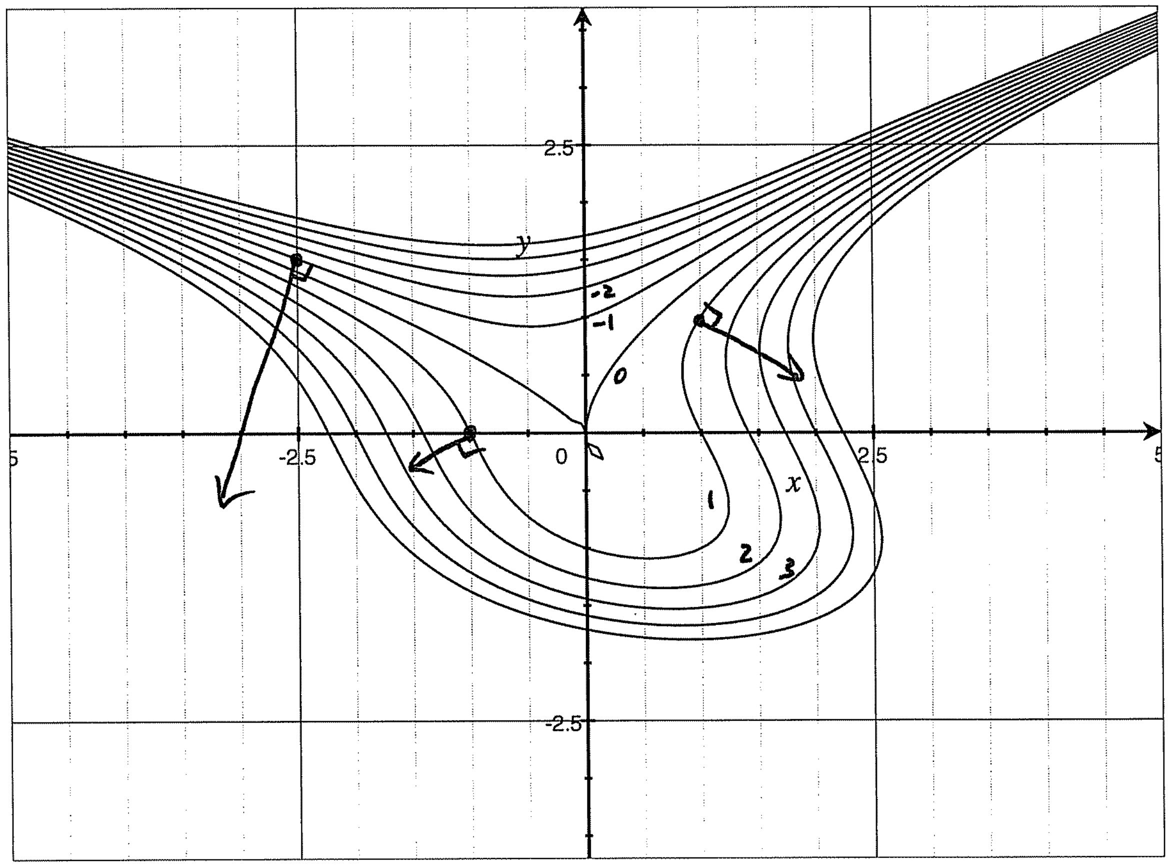
$$D = f_{xx} f_{yy} - f_{xy}^2 = 4 - k^2.$$

When $k = \pm 2$, we have $D = 0$, so we have no information. When $k > 2$ or $k < -2$, we have $D < 0$, so that this is a saddle point. When $-2 < k < 2$, we have that $D > 0$, so that this point is either a max or min. Because $f_{xx} > 0$, it must be a min.

4. The following is a contour map for the function $f(x,y)$.

(a) Estimate the value of f at the point $(1, 1.5)$.

This point lies on the level curve corresponding to $f(x,y) = -1$. Thus $f(1, 1.5) = -1$.



(b) Is $f_x(1, 1.5)$ positive, negative, or approximately zero?

It is positive, because moving from $(1, 1.5)$ in the positive x-direction represents an increase in function values.

(c) What about $f_y(1, 1.5)$?

It is negative, because moving from $(1, 1.5)$ in the positive y-direction represents a decrease in function values.

(d) What about $D_{\mathbf{u}}(1, 1.5)$, where $\mathbf{u} = \langle 1, 1 \rangle$?

This is approximately zero, or possibly a bit negative. The level curve at the point $(1, 1.5)$ is almost tangent to the direction $\langle 1, 1 \rangle$, representing no change in function values.

(e) Which is larger (in absolute value), $f_y(1, 1)$ or $f_y(1, 1.5)$?

The larger of the two (in absolute value) is $f_y(1, 1.5)$, because the level curves there in the y-direction are closer together, representing a steeper ascent in function values in that direction.

(f) At all marked points, sketch the gradient vector for the function, keeping in mind their relative lengths.

See graph.

5. You are a wasp flying around a room. The temperature at the point in the room with coordinates (x, y, z) is given by $T(x, y, z) = x^2y + y^2z - z^2x$. You are currently at the point $(-1, 0, 1)$, where the temperature is $T(-1, 0, 1) = 1$.

(a) Find the direction in which you should fly in order to increase temperature as fast as possible.

You should fly in the direction of the gradient, which is

$$\nabla T(-1, 0, 1) = \langle 2xy - z^2, x^2 + 2yz, y^2 - 2zx \rangle|_{(-1,0,1)} = \langle -1, 1, 2 \rangle.$$

(b) Find the equation for the plane tangent to the level surface $T(x, y, z) = 1$ at the point $(-1, 0, 1)$.

The gradient vector found above is a normal vector for this plane, so the plane is described by

$$-1(x + 1) + 1(y - 0) + 2(z - 1) = 0.$$

(c) What will the rate of change of temperature be if you fly in the direction of the vector $\langle 1, 1, 1 \rangle$?

This is a directional derivative, so we first normalize the direction, obtaining $\mathbf{u} = \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle$. We then compute

$$D_{\mathbf{u}}(-1, 0, 1) = \nabla T \cdot \mathbf{u} = \langle -1, 1, 2 \rangle \cdot \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle = \frac{2}{\sqrt{3}}.$$